Soft-Pion Theorem for Hard Near-Threshold Pion Production

P. V. Pobylitsa,[1,2] M. V. Polyakov,[1,2] and M. Strikman[1,3]
[1]*Petersburg Nuclear Physics Institute, 188350 Gatchina, Russia*
[2]*Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany*
[3]*Pennsylvania State University, University Park, Pennsylvania 16802*
(Received 6 February 2001; published 20 June 2001)

We prove a new soft-pion theorem for the near-threshold pion production by a hard electromagnetic probe. This theorem relates various near-threshold pion-production amplitudes to the nucleon distribution amplitudes. The new soft-pion theorem is in good agreement with the SLAC data for the structure function $F_2^p(W, Q^2)$ for $W^2 \leq 1.4\ \text{GeV}^2$ and $9 \leq Q^2 \leq 30.7\ \text{GeV}^2$.

DOI: 10.1103/PhysRevLett.87.022001 PACS numbers: 12.38.Lg, 11.30.Rd, 13.60.Fz, 13.60.Le

The amplitudes of the pion production near the threshold by the electromagnetic probe

$$\gamma^* + N \to \pi + N' \tag{1}$$

at a not too large virtuality (Q^2) of the photon $Q^2 \ll \Lambda^3/m_\pi$ ($\Lambda \sim 1$ GeV is a typical hadronic scale) can be expressed in terms of various nucleon form factors with the help of the soft-pion theorem (SPT) [1–3]. For virtualities $Q^2 \sim \Lambda^3/m_\pi$ and larger this SPT does not work, because the SPT of [1–3] corresponds to first taking the chiral limit ($m_\pi \to 0$) while Q^2 is kept fixed. To study the parametric domain $m_\pi Q^2 \gg \Lambda^3$ we consider the opposite sequence of the limits: $Q^2 \to \infty$ first and $m_\pi \to 0$ second. Since the two limits do not commute, in this way we derive a new "hard-soft" pion theorem (hSPT) for the reaction (1) in the near-threshold region and in the Q^2 kinematics complementary to one of Refs. [1–3], namely, $m_\pi Q^2 \gg \Lambda^3$. Our main tool is the QCD factorization theorem for exclusive processes [4,5] (for a recent review and comprehensive list of references, see, e.g., [6]). It allows us to express the pion-production amplitude at large virtuality in terms of the distribution amplitudes (DAs) of the nucleon and of the low-mass πN system. These nonperturbative objects correspond to the minimal three quark ($3q$) Fock component of the nucleon and πN systems [7]. We derive a hSPT to relate the corresponding distribution amplitude of the πN system (we call it πN DA) to the nucleon distribution amplitude. This hSPT is valid for the limit when the mass of the πN system (denoted as W) is close to the threshold $W_{\text{th}} = M_N + m_\pi$. The derivation of the similar theorem for DA of the two-pion system near threshold can be found in [8,9].

The physical picture of the near-threshold production of pion by a hard electromagnetic probe is as follows. At large Q^2 the emission of the soft pion from the initial state contributes only to large invariant masses W. The emission from the hard interaction part is a higher twist in Q^2. Hence the emission occurs solely in the final state when a small $3q$ system produced in the hard interaction expands to a large enough configuration. At this point we are dealing with a soft-pion emission and can apply corresponding near-threshold chiral theory relations.

It follows from the QCD factorization theorem [4,5] that the transition matrix element $A(\gamma^* N \to \pi N')$ at large Q^2 can be written as (up to the power suppressed terms)

$$A(\gamma^* N \to \pi N') = \int dx\, dy\, \Phi^*_{\pi N'}(x) T(x,y) \Phi_N(y), \tag{2}$$

where $T(x,y)$ is the hard part of the process computable in perturbative QCD. The functions $\Phi_N(y)$, $\Phi_{\pi N'}(x)$ are distribution amplitudes (light cone wave functions) of the nucleon and of the low-mass $\pi N'$ system. The DA of the nucleon $\Phi_N(y)$ also enters the QCD description of the nucleon form factor and it is the subject of intensive studies. The distribution amplitude of the πN system is a straightforward generalization of the baryon DAs. Many general properties of these objects are similar to those of the two-pion distribution amplitude which were extensively studied in Ref. [9]. They will be discussed elsewhere.

We focus here on the soft-pion theorem for πN DAs. Using the general soft-pion theorem (see, e.g., [10]) we can write

$$\langle \pi^a(k) N_f(p,S)|O|0\rangle = \frac{i}{f_\pi}\langle N_f(p,S)|[Q_5^a, O]|0\rangle + \frac{i g_A}{4 f_\pi (p\cdot k)} \sum_{S',f'} \bar{u}(p,S)\not{k}\gamma_5 \tau^a_{ff'} u(p,S')\langle N_{f'}(p,S')|O|0\rangle. \tag{3}$$

Here $f_\pi = 93$ MeV is the pion decay constant and $g_A \approx 1.25$ is the axial charge of the nucleon. The operator O is the trilocal quark operator of twist-3 in which quark fields are separated by a light cone distance, in the complete analogy with the definition of the baryon DA; see, e.g., [11]. The operator Q_5^a is the operator of axial charge, so that the calculation of the commutator $[Q_5^a, O]$ reduces to the chiral rotation of operator O. The second term on the right-hand side (rhs) of Eq. (3) corresponds to the chiral singularity due to the nucleon pole in the graph shown in Fig. 1. The contribution of this diagram is strongly

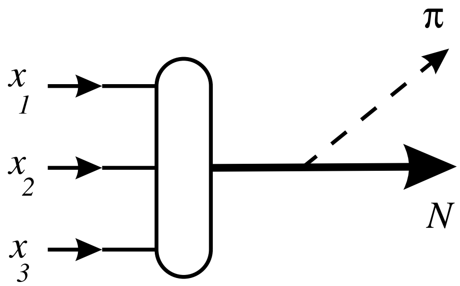

FIG. 1. Nucleon pole contribution to the soft-pion theorem for generalized πN distribution amplitudes.

suppressed for $W - W_{\rm th} \ll m_\pi$, but for $W - W_{\rm th} \sim m_\pi$ it becomes significant; see Eqs. (18) and (19).

Let us start from the calculation of the first (commutator) term in Eq. (3). Since the chiral rotation of the trilocal quark operator O does not change its twist Eq. (3) allows us to express generalized πN DAs at the threshold in terms of nucleon DAs.

We write the nucleon DA in terms of functions $\phi_S(x)$ and $\phi_A(x)$ which are symmetric and antisymmetric, respectively, under $x_1 \leftrightarrow x_3$ (1 and 3 are quarks with parallel helicities) [4,12]. For the proton we have

$$|p\uparrow\rangle = \frac{\phi_S(x)}{\sqrt{6}}|2u_\uparrow d_\downarrow u_\uparrow - u_\uparrow u_\downarrow d_\uparrow - d_\uparrow u_\downarrow u_\uparrow\rangle + \frac{\phi_A(x)}{\sqrt{2}}|u_\uparrow u_\downarrow d_\uparrow - d_\uparrow u_\downarrow u_\uparrow\rangle . \tag{4}$$

The distribution amplitude for neutron can be obtained from the above expression by the replacement $u \leftrightarrow d$.

Applying the general soft-pion theorem (3) we express the distribution amplitudes of various πN systems at the threshold in terms of the nucleon DAs $\phi_S(x)$ and $\phi_A(x)$:

$$|p\uparrow\pi^0\rangle = \frac{\phi_S(x)}{2\sqrt{6}f_\pi}|6u_\uparrow d_\downarrow u_\uparrow + u_\uparrow u_\downarrow d_\uparrow + d_\uparrow u_\downarrow u_\uparrow\rangle - \frac{\phi_A(x)}{2\sqrt{2}f_\pi}|u_\uparrow u_\downarrow d_\uparrow - d_\uparrow u_\downarrow u_\uparrow\rangle , \tag{5}$$

$$|n\uparrow\pi^+\rangle = \frac{\phi_S(x)}{\sqrt{12}f_\pi}|2u_\uparrow d_\downarrow u_\uparrow - 3u_\uparrow u_\downarrow d_\uparrow - 3d_\uparrow u_\downarrow u_\uparrow\rangle - \frac{\phi_A(x)}{2f_\pi}|u_\uparrow u_\downarrow d_\uparrow - d_\uparrow u_\downarrow u_\uparrow\rangle . \tag{6}$$

The DAs of the neutral πN systems can be obtained by the trivial replacement $u \leftrightarrow d$.

Now we can compute the threshold amplitudes $A(\gamma^* N \to \pi N')$ at large Q^2 combining the factorization theorem (2) with the expressions for πN DAs (5),(6). The technique of calculations of the hard part $T(x,y,Q^2)$ is standard and can be found, e.g., in Refs. [4,11]. We give here only the final results using notations of Refs. [4,12,13]. Generically for the transitions $N \to \pi N'$ we can write

$$Q^4 A(\gamma^* N \to \pi N')|_{\rm th} = \frac{(16\pi\alpha_s)^2}{9f_\pi}\int [dx\,dy] \times \sum_{i,j=S,A} C_{ij}^{N\to\pi N'}(x,y)\phi_i(x) \times \phi_j(y) . \tag{7}$$

Here $[dx] = dx_1 dx_2 dx_3\ \delta(1 - x_1 - x_2 - x_3)$ and coefficient functions $C_{ij}(x,y)$ are given in the following table:

$N \to \pi N'$	C_{SS}	$C_{SA} = C_{AS}$	C_{AA}
$p \to \pi^0 p$	$\frac{23}{9}T_1 - \frac{8}{9}T_2$	$\frac{1}{\sqrt{3}}T_1$	$-\frac{13}{9}(T_1 + 2T_2)$
$p \to \pi^+ n$	$\frac{\sqrt{2}}{9}(11T_1 + 4T_2)$	$-\sqrt{\frac{2}{3}}T_1$	$-\sqrt{\frac{2}{3}}(T_1 + 2T_2)$
$n \to \pi^0 n$	$\frac{13}{9}(T_1 - T_2)$	$\frac{1}{\sqrt{3}}T_1$	$\frac{1}{3}(T_1 + T_2)$
$n \to \pi^- p$	$-\frac{\sqrt{2}}{9}(T_1 - T_2)$	$\sqrt{\frac{2}{3}}T_1$	$-\sqrt{\frac{2}{3}}(T_1 - T_2)$

with the standard functions [4]

$$T_1 = \frac{1}{x_3(1-x_1)^2 y_3(1-y_1)^2} + \frac{1}{x_2(1-x_1)^2 y_2(1-y_1)^2} - \frac{1}{x_2x_3(1-x_3)y_2y_3(1-y_1)} , \tag{8}$$

$$T_2 = \frac{1}{x_1x_3(1-x_1)y_1y_3(1-y_3)} . \tag{9}$$

Expressions for various nucleon form factors in the same notations can be found, e.g., in Refs. [4,12,13].

Using Eq. (7) one can express the near-threshold structure functions $F_2^{p,n}(W,Q^2)$ and various differential cross sections for particular πN channels in terms of the nucleon DAs. The *same* DAs appear in the QCD factorization theorem for the nucleon form factors at large Q^2. We find that in the case of the symmetric form of the nucleon DA

$$\phi_S(x) \quad \text{is arbitrary,} \qquad \phi_A(x) = 0 , \tag{10}$$

one can describe the near-threshold pion production directly in terms of the nucleon form factors without specifying the nucleon DA; see below. Certainly the nucleon DA can have a nonzero asymmetric component $\phi_A(x)$. Actually our general Eq. (7) can be applied to any specific model of the nucleon DA (see, e.g., [14]) to compare the model predictions with the experiment. In the case of symmetric nucleon DA the amplitude of the process (1) can be expressed in terms of nucleon magnetic form factors $[G_{MN}(Q^2)]$ as follows:

$$A(\gamma^* p \to \pi^0 p)|_{\rm th} = -\frac{1}{f_\pi}\left(\frac{5}{6}G_{Mp} - \frac{4}{3}G_{Mn}\right), \tag{11}$$

$$A(\gamma^* p \to \pi^+ n)|_{\rm th} = \frac{1}{\sqrt{2}f_\pi}\left(\frac{5}{3}G_{Mp} + \frac{4}{3}G_{Mn}\right), \tag{12}$$

$$A(\gamma^* n \to \pi^0 n)|_{\rm th} = -\frac{13G_{Mn}}{6f_\pi} , \tag{13}$$

$$A(\gamma^* n \to \pi^- p)|_{\rm th} = \frac{G_{Mn}}{3\sqrt{2}f_\pi} . \tag{14}$$

Since these results are based on the symmetric form (10) for the nucleon distribution amplitude, deviations from these equations would allow one to probe directly the asymmetric part of the nucleon distribution amplitude and check the validity of the leading twist description of the nucleon form factors.

The data on the structure function $F_2^p(W,Q^2)$ in the near-threshold region were obtained in 1994 in the SLAC experiment E136 [15] for a wide range of $Q^2 = 7-30.7\ \mathrm{GeV}^2$. Here we make the first comparison of the data with the hSPT predictions.

To compute $F_2^p(W,Q^2)$ for $W - W_{\mathrm{th}} \lesssim m_\pi$ we combine the strictly threshold amplitude (7) with the contribution of the last term from Eq. (3) and obtain

$$F_2^p(W,Q^2) = \frac{Q^2\beta(W)}{(4\pi)^2}\left[\sum_{X=p\pi^0,n\pi^+}|A(p\to X)|_{\mathrm{th}}^2 + \frac{3g_A^2G_{Mp}^2(Q^2)\beta^2(W)W^4}{4f_\pi^2(W^2-M_N^2+m_\pi^2)^2} + O\left(\frac{m_\pi}{\Lambda}\right)\right], \tag{15}$$

where

$$\beta(W) = \sqrt{1-\frac{(M_N+m_\pi)^2}{W^2}}\sqrt{1-\frac{(M_N-m_\pi)^2}{W^2}}. \tag{16}$$

The first term on the rhs of Eq. (15) corresponds to strictly threshold amplitude (7). The second term takes into account the emission of the pion from the outgoing nucleon (Fig. 1) with the amplitude given by the second term on the rhs of Eq. (3). The latter contribution vanishes exactly at the threshold but gives a parametrically unsuppressed contribution for $W - W_{\mathrm{th}} \sim m_\pi$. Note that the second term in Eq. (15) corresponds to the πN system in the P wave; therefore it can be separated from the first (S-wave) term by considering the angular distributions in πN system.

The data of the E136 experiment [15] are consistent with the factorized form

$$F_2^p(W,Q^2) = F(Q^2)G(W), \tag{17}$$

with $F(Q^2) \propto 1/Q^6$ for $Q^2 \geq 8\ \mathrm{GeV}^2$ which is exactly the scaling behavior following from Eq. (15). We also found that Eq. (15) provides a good description of the W dependence of the E136 data for $W \leq 1.2$ GeV though the predicted W dependence is somewhat different from the $G(W) \propto W^2 - W_{\mathrm{th}}^2$ fit of [15]. However, the resolution of E136 is not sufficient to distinguish between the two forms; see Fig. 2. The hSPT (15) also predicts the scaling behavior $\sim 1/Q^6$ which is confirmed by the E136 data [15].

Thus we reproduce several features of the data without using any specific nucleon wave functions. To make a first quantitative comparison of the soft-pion theorem prediction for the absolute value of F_2^p we use the symmetric form for the nucleon distribution amplitude (10). Inserting the expressions (11) and (12) for the threshold amplitudes into Eq. (15) and using the analogous expression for the neutron structure function, we obtain the following hSPT for the structure functions in the case of the symmetric form (10) for the nucleon DA

$$F_2^p(W,Q^2) = \frac{Q^2\beta(W)}{(4\pi f_\pi)^2}\left[\frac{25}{12}G_{Mp}^2(Q^2) + \frac{8}{3}G_{Mn}^2(Q^2) + \frac{3g_A^2G_{Mp}^2(Q^2)\beta^2(W)W^4}{4(W^2-M_N^2+m_\pi^2)^2} + O\left(\frac{m_\pi}{\Lambda}\right)\right], \tag{18}$$

$$F_2^n(W,Q^2) = \frac{Q^2\beta(W)}{(4\pi f_\pi)^2}\left[\frac{19}{4}G_{Mn}^2(Q^2) + \frac{3g_A^2G_{Mn}^2(Q^2)\beta^2(W)W^4}{4(W^2-M_N^2+m_\pi^2)^2} + O\left(\frac{m_\pi}{\Lambda}\right)\right]. \tag{19}$$

Now we can compare our results based on the hSPT with the E136 data [15]. In Fig. 2 we show the E136 data on the W^2 dependence (for $W^2 \leq 1.4\ \mathrm{GeV}^2$ to ensure that $W - W_{\mathrm{th}} \lesssim m_\pi$) of the $Q^6F_2^p(W,Q^2)$ at various values of $Q^2 \geq 9\ \mathrm{GeV}^2$ compared with the prediction of the hSPT Eq. (18) for three values of $Q^2 = 10, 20, 30\ \mathrm{GeV}^2$. For $G_{MN}(Q^2)$ we used parametrizations given in Ref. [16]. We see that our predictions are consistent with experimental data, although the accuracy of the data is not sufficient to make a detailed comparison. Hence we also consider the integrated quantity $\int_{\mathrm{th}}^{1.4} dW^2\, Q^6F_2^p(W,Q^2)$ which was measured in [15] with a better accuracy and was found to be practically constant for $Q^2 \geq 9\ \mathrm{GeV}^2$. The experimental value of the integral is in good agreement with our result (18):

$$\int_{\mathrm{th}}^{1.4} dW^2\, Q^6F_2^p(W,Q^2) = \begin{cases} 0.10 \pm 0.02\ \mathrm{GeV}^8 & (\mathrm{E136}), \\ 0.11 \pm 0.02\ \mathrm{GeV}^8 & (\mathrm{hSPT}). \end{cases} \tag{20}$$

For the theoretical analysis we used as input the following values for the nucleon form factors $Q^4G_{Mp}(Q^2) = 1.0 \pm 0.1\ \mathrm{GeV}^4$ obtained at $Q^2 \geq 10\ \mathrm{GeV}^2$ [17] and $Q^4G_{Mn}(Q^2) = -(0.5 \pm 0.1)\ \mathrm{GeV}^4$ extracted from Ref. [18] at $Q^2 \approx 10\ \mathrm{GeV}^2$ [19]. For the summary of the current information on various baryon form factors at large momentum transfer, see, e.g., [20]. Note that the contribution of the P-wave term in Eq. (18) is relatively small (about 20%), so that the main part of the hSPT value (20) is due to the strictly threshold contribution (S wave) in Eq. (15). With the same set of parameters we predict $\int_{\mathrm{th}}^{1.4} dW^2\, Q^6F_2^n(W,Q^2) = 0.05 \pm 0.02\ \mathrm{GeV}^8$. Note that for the nucleon DAs which fit $G_{Mp}(Q^2)$, $G_{Mn}(Q^2)$ at $Q^2 \geq 10\ \mathrm{GeV}^2$ (such DA significantly differ from the asymptotic one; see, e.g., [12,13]) the F_2^n/F_2^p ratio near

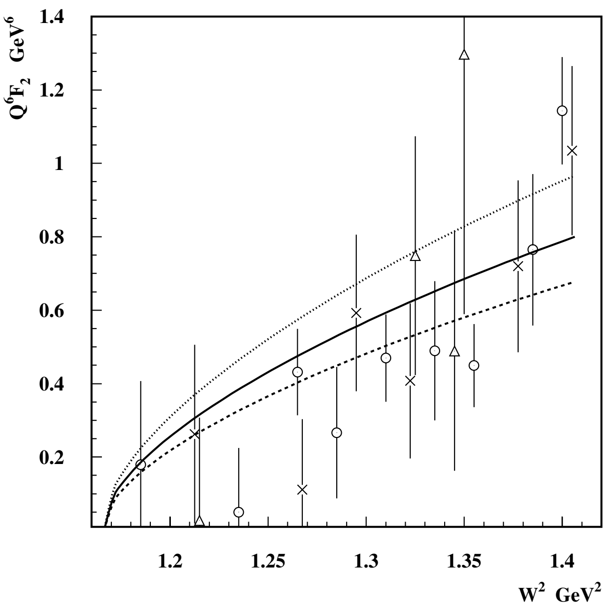

FIG. 2. Values of $F_2^p(W, Q^2)$ scaled by Q^6 as a function of W^2. The data of the E136 experiment are at average Q^2 values of 9.4, 11.8 (×), 15.5, 19.2 (○), 23, 26, and 31 (△) GeV^2. The theoretical predictions of the hSPT (18) at $Q^2 =$ 10, 20, 30 GeV^2 are given by dotted, solid, and dashed lines respectively.

threshold is much smaller than the asymptotic value of 52/37 which follows from Eq. (7) with the asymptotic distribution amplitude $\phi(x) \propto x_1 x_2 x_3$. Thus this ratio is extremely sensitive to the deviations of the nucleon DA from the asymptotic form. Therefore measurements of the neutron structure function in the near-threshold region would considerably constrain the form of the nucleon distribution amplitude.

In this Letter we derived a new soft-pion theorem for the threshold pion production by a hard electromagnetic probe, i.e., with the probe of virtuality $Q^2 \gg \Lambda^2$ ($\Lambda \sim 1$ GeV is a typical hadronic scale). This new hSPT allows us to express the pion-production amplitudes in terms of the distribution amplitudes of the nucleon. The latter enter the description of various nucleon form factors at large momentum transfer. These new relations give a possibility to constrain further the nucleon distribution amplitude using data on threshold inelastic electron scattering from the nucleon at high momentum transfer.

Using a generic symmetric model for the nucleon DAs we demonstrated that various observables for near-threshold pion production at high momentum transfer are sensitive to the parameters of nucleon DAs. This shows that the near-threshold pion production by a hard electromagnetic probe is a new valuable source of information about nucleon distribution amplitudes. Studies with a broader range of models of nucleon DAs will be presented elsewhere.

Our analysis was restricted to the leading twist QCD contributions. The application of the methods developed here to the models for soft contributions to the baryon form factors (see a review in [21]) would allow one to derive predictions of these models for hard near-threshold pion production. This might be an exciting possibility to use hSPT to discriminate between soft and hard mechanisms for high momentum transfer reactions. The study of the discussed processes should be feasible at the top of the current JLab energies and should be one of the high priorities of JLab at 12 GeV.

[1] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954); Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962); Y. Nambu and E. Shrauner, Phys. Rev. **128**, 862 (1962).
[2] A. I. Vainshtein and V. I. Zakharov, Nucl. Phys **B36**, 589 (1972).
[3] S. Scherer and J. H. Koch, Nucl. Phys. **A534**, 461 (1991).
[4] P. Lepage and S. Brodsky, Phys. Rev. Lett. **43**, 545 (1979); **43**, 1625(E) (1979); P. Lepage and S. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[5] A. V. Efremov and A. V. Radyushkin, Theor. Math. Phys. **44**, 774 (1981).
[6] S. J. Brodsky, "Exclusive Processes in Quantum Chromodynamics and the Light-Cone Fock Representation," SLAC Report No. SLAC-PUB-8649.
[7] Obviously, a small size $3q$ system has comparable overlaps with a nucleon, N^*, Δ, πN, etc. states.
[8] O. Teryaev, in *Proceedings of the XIVth International Seminar on High Energy Physics Problems* (JINR, Dubna, 1998).
[9] M. V. Polyakov, Nucl. Phys. **B555**, 231 (1999).
[10] S. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[11] V. Chernyak and A. Zhitnitsky, Phys. Rep. **112**, 173 (1984).
[12] C. E. Carlson and J. L. Poor, Phys. Rev. D **34**, 1478 (1986).
[13] C. E. Carlson, M. Gari, and N. G. Stefanis, Phys. Rev. Lett. **58**, 1308 (1987).
[14] V. L. Chernyak and I. R. Zhitnitsky, Nucl. Phys. **B246**, 52 (1984); M. Gari and N. G. Stefanis, Phys. Lett. B **175**, 462 (1986); Phys. Rev. D **35**, 1074 (1987); V. L. Chernyak, A. A. Ogloblin, and I. R. Zhitnitsky, Sov. J. Nucl. Phys. **48**, 536 (1988); N. Stefanis and M. Bergmann, Phys. Rev. D **47**, 3685 (1993); R. Jakob, P. Kroll, M. Schurmann, and W. Schweiger, Z. Phys. A **347**, 109 (1993); J. Bolz and P. Kroll, Z. Phys. A **356**, 327 (1996).
[15] P. Bosted *et al.,* Phys. Rev. D **49**, 3091 (1994).
[16] P. E. Bosted, Phys. Rev. C **51**, 409 (1995).
[17] R. G. Arnold *et al.,* Phys. Rev. Lett. **57**, 174 (1986); A. F. Sill *et al.,* Phys. Rev. D **48**, 29 (1993).
[18] S. Rock *et al.,* Phys. Rev. D **46**, 24 (1992).
[19] Note that in perturbative QCD $Q^4 G_{MN}(Q^2)$ should logarithmically decrease with Q^2. The E134 data support such a trend for $Q^2 > 10$ GeV^2. This would imply a similar decrease for the integral (20), for example, by a factor $\sim$1.5 between $Q^2 = 10$ and 31 GeV^2. The data [15] do not contradict to such a decrease, though the errors become large for the data points at the highest Q^2.
[20] P. Stoler, Phys. Rep. **226**, 103 (1993); G. Sterman and P. Stoler, Annu. Rev. Nucl. Part. Sci. **43**, 193 (1997).
[21] A. V. Radyushkin, Few-Body Syst. Suppl. **11**, 57 (1999).