Dynamic Long-Term Anticipation of Chaotic States

Henning U. Voss*
Department of Physics, University of Freiburg, Freiburg, Germany
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Introducing a short time delay into the coupling of two synchronizing chaotic systems, it was shown recently that the driven system may anticipate the driving system in real time. Augmenting the phase space of the driven system, we accomplish anticipation times that are multiples of the coupling delay time and exceed characteristic time scales of the chaotic dynamics. The stability properties of the associated anticipatory synchronization manifold in certain cases turn out to be the same as for identically synchronizing oscillators.

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Since the discovery of chaotic dynamics in continuous nonlinear dissipative systems some decades ago [1] this phenomenon has attracted an enormous research activity in different areas of science. On the one hand, it was found that seemingly simple deterministic systems can exhibit rather complex solutions, but on the other hand, coupled chaotic systems may exhibit ordered collective behavior by the emergence of synchronization between them [2]. In this case the time evolution becomes restricted to a dynamically invariant subspace of the chaotic phase space, the synchronization manifold. In other words, some degrees of freedom of the joint system are eliminated, leading to a reduction of complexity. Furthermore, since the availability of methods for chaos control [3], deterministic chaos has lost even more of its unpredictability.

Recently, by introducing bitemporal synchronization manifolds which comprise two different time points, for unidirectionally coupled systems the synchronization behavior was generalized in the following sense: The driving system's state is replicated not instantaneously but *anticipated* by the driven system [4,5], without affecting the driving system again. This phenomenon causes an elimination of degrees of freedom in cases that have not been considered yet and may be highly desirable in technical applications. For example, it was already reproduced in realistic numerical simulations of coupled chaotic semiconductor lasers [6]. However, for chaotic systems without an intrinsic time delay, the maximum stably attainable anticipation time turned out to be much shorter than the characteristic time scales of the system's dynamics.

In this Letter we show that stability can be enhanced considerably by using chains of oscillators. As a result, we accomplish anticipation times larger than characteristic time scales of the system's dynamics, thus introducing a novel way of reducing unpredictability of chaotic dynamics. The outline of this Letter is as follows: After briefly reviewing the phenomena of identical and anticipating synchronization, we introduce oscillator chains to augment the phase space. This leads to a considerable increment of the maximum attainable anticipation time. Then we show that in certain cases these chains have exactly the same stability properties as chains of identically synchronizing oscillators, a result important in view of disturbed or nonidentical systems.

We start with the well known phenomenon of *identical* chaotic synchronization: Consider a dynamical system given by the vector field

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad [\mathbf{x} \in \mathbb{R}^n], \tag{1}$$

which is coupled to an identical system with state vector $\mathbf{y}$ via

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + \mathbf{k}(\mathbf{x} - \mathbf{y}) \qquad [\mathbf{y} \in \mathbb{R}^n]. \tag{2}$$

The *transversal system* with the state variable $\mathbf{\Delta} := \mathbf{x} - \mathbf{y}$ then evolves as

$$\dot{\mathbf{\Delta}} = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) - \mathbf{k}(\mathbf{\Delta}). \tag{3}$$

Often, the coupling is considered to be linear; i.e., $\mathbf{k}(\mathbf{\Delta})$ becomes an $n \times n$ matrix that is multiplied with the transversal state. In the following, we do not notationally distinguish the function and matrix $\mathbf{k}$. In any case, it is assumed that $\mathbf{k}(\mathbf{0}) = \mathbf{0}$. If $\mathbf{k}$ is linear and $\mathbf{f}$ is a polynomial up to second order, the then linear nonautonomous transversal system is

$$\dot{\mathbf{\Delta}} = [\mathbf{g}(\mathbf{x}, \mathbf{y}) - \mathbf{k}]\mathbf{\Delta}, \tag{4}$$

with $\mathbf{g}$ an $n \times n$ matrix depending linearly on $\mathbf{x}$ and $\mathbf{y}$. If the fixed point $\mathbf{\Delta} = \mathbf{0}$ is globally asymptotically stable, i.e., $\lim_{t\to\infty} \|\mathbf{\Delta}(t)\| = 0$ for any initial condition $\mathbf{\Delta}(0)$, the two systems will exhibit synchronous behavior after some transient time [7]. Then the dynamics is restricted to the subspace $\mathbf{x} = \mathbf{y}$ which is usually called the *synchronization manifold* [8] of the coupled systems. On this manifold, the coupling term vanishes, and the driving system (1) and driven system (2) become identical. There are several examples for chaotic systems in which identical synchronization was observed, like the Rössler [9] and Lorenz [1] oscillators. Figures 1(a) and 1(b) depict the approach to the synchronization manifold for two coupled Rössler oscillators, where the driven oscillator is given by $\dot{y}_1 = -y_2 - y_3 + k(x_1 - y_1)$, $\dot{y}_2 = y_1 + a y_2$, and $\dot{y}_3 = b + y_3(y_1 - c)$. With this scalar coupling the transversal system has the form of Eq. (4) [4].

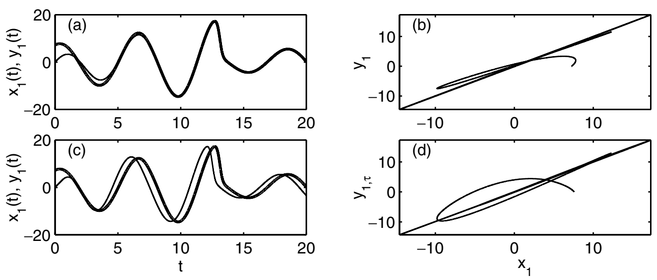

FIG. 1. (a) Numerically simulated time series $x_1(t)$ (bold line) and $y_1(t)$ for two coupled Rössler oscillators and (b) their approach to the manifold $\mathbf{x} = \mathbf{y}$. The coupling strength $k = 0.8$ is used. (c) The same for anticipatory coupling with $\tau = 0.6$, i.e., the coupling term $0.8(x_1 - y_{1,\tau})$ is used in Eq. (5). The driven system's trajectory is shifted 0.6 time units to the left, thus anticipating the chaotic driver. This is also seen in the approach to the anticipatory synchronization manifold $\mathbf{x} = \mathbf{y}_\tau$ (d). (In all simulations the parameters of the Rössler oscillators are $a = 0.15$, $b = 0.2$, and $c = 10$. The other components of the two synchronization manifolds do not qualitatively differ from the shown first components.)

To construct a system that anticipates the dynamics of Eq. (1), a time delay $\tau \geq 0$ is introduced by modifying the coupling term in Eq. (2) to $\mathbf{k}(\mathbf{x} - \mathbf{y}_\tau)$, where $\mathbf{y}_\tau := \mathbf{y}(t - \tau)$. This so introduced "memory" yields the system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + \mathbf{k}(\mathbf{x} - \mathbf{y}_\tau). \tag{5}$$

Note that the driving system is not affected and only past values of the driven system are needed. This concept allows for the definition of a bitemporal *anticipatory synchronization manifold*

$$\mathbf{x} = \mathbf{y}_\tau, \tag{6}$$

on which the state of the driven system, $\mathbf{y}$, anticipates the driving system's state $\mathbf{x}$. This is easily seen by a time shift in Eq. (6) to yield $\mathbf{x}(t + \tau) = \mathbf{y}(t)$. If now the transversal system state is defined accordingly as $\boldsymbol{\Delta}^{(\tau)} := \mathbf{x} - \mathbf{y}_\tau$, for its time evolution one finds immediately

$$\dot{\boldsymbol{\Delta}}^{(\tau)} = \mathbf{g}(\mathbf{x}, \mathbf{y}_\tau)\boldsymbol{\Delta}^{(\tau)} - \mathbf{k}\boldsymbol{\Delta}^{(\tau)}_\tau, \tag{7}$$

where again $\boldsymbol{\Delta}^{(\tau)}_\tau := \boldsymbol{\Delta}^{(\tau)}(t - \tau)$. Obviously, for *any* time delay τ, $\boldsymbol{\Delta}^{(\tau)} = \mathbf{0}$ is a fixed point of this system, and, therefore, the anticipatory synchronization manifold (6) exists.

As an example, we use anticipatory coupling of two coupled Rössler oscillators again. The transversal system becomes

$$\dot{\boldsymbol{\Delta}}^{(\tau)} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ x_3 & 0 & y_{1,\tau} - c \end{pmatrix} \boldsymbol{\Delta}^{(\tau)} - \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\Delta}^{(\tau)}_\tau. \tag{8}$$

The approach of the trajectories to the now anticipatory synchronization manifold is depicted in Figs. 1(c) and 1(d).

Whereas the *existence* of the anticipatory manifold $\mathbf{x} = \mathbf{y}_\tau$ could easily be proved for arbitrary τ, the particular value for τ for which *stability* of this manifold breaks down is much harder to find. Even in the case of identical synchronization, stability has to be examined using numerical simulations, for example, by checking the transversal Lyapunov exponents [8,10] for negativity [11] (unless a Lyapunov function for the transversal system can be found [12,13]). Therefore, we argue only for the existence of a small τ for which the manifold $\mathbf{x} = \mathbf{y}_\tau$ is linearly stable: Formally, the stability of the fixed point $\boldsymbol{\Delta}^{(\tau)} = \mathbf{0}$ is determined by the (time-dependent) roots of the characteristic equation [14] of Eq. (7),

$$\det\{\lambda \mathbf{I} - [\mathbf{g}(\mathbf{x}, \mathbf{y}_\tau) - \mathbf{k}e^{-\lambda\tau}]\} = 0. \tag{9}$$

For stability of identical synchronization ($\tau = 0$), contrary to ordinary differential equations with constant coefficients, it may not be necessary that all roots are located in the open left half complex plane for all possible values of $(\mathbf{x}, \mathbf{y})$. This would be far too conservative a condition for stability, and it is violated clearly in the examples considered here. (For these reasons it was proposed in Ref. [15] to consider the *averaged* eigenvalues of the Jacobian along a driving system's trajectory as a sufficient condition for synchronization.) For $\tau > 0$, this now transcendental equation may have a completely different spectrum of eigenvalues, but for a sufficiently small τ it can be assumed quite generally that the roots are disturbed only weakly and still can be considered good approximations of the roots with $\tau = 0$. For these reasons, we make the following conjecture: If $\boldsymbol{\Delta} = \mathbf{0}$ is a stable fixed point of Eq. (4), for most systems there exists a $\tau_0 > 0$ such that for all $\tau < \tau_0$ $\boldsymbol{\Delta}^{(\tau)} = \mathbf{0}$ remains to be a stable fixed point of Eq. (7). This conjecture is backed up by numerical simulations [4]; in case of coupled Rössler systems, numerical computations of the largest transversal Lyapunov exponent reveal that it remains negative in a broad area of the (k, τ)-parameter space.

To increase the maximum possible anticipation time τ, we augment the phase space in the following way: If for $\tau = \tau_0$ the anticipatory synchronization manifold of system (5) is stable, to yield a manifold with $\tau = m\tau_0$ ($m = 2, 3, \ldots$), we define a chain of m systems each with state vector $\mathbf{y}_i \in \mathbb{R}^n$. Defining the augmented state vector as $\mathbf{Y} := (\mathbf{y}_1^T, \ldots, \mathbf{y}_m^T)^T \in \mathbb{R}^{mn}$, the complete driven system becomes

$$\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y}) + \mathbf{K}(\mathbf{X} - \mathbf{Y}_{\tau_0}), \tag{10}$$

with $\mathbf{X} := (\mathbf{x}^T, \mathbf{y}_1^T, \ldots, \mathbf{y}_{m-1}^T)^T$, $\mathbf{F} := (\mathbf{F}_1^T, \ldots, \mathbf{F}_m^T)^T$, $\mathbf{F}_i$: $\mathbf{y}_i \mapsto \mathbf{f}(\mathbf{y}_i)$ ($i = 1, \ldots, m$), analogous with $\mathbf{K}$. In the case of synchronization of systems (1) and (5), we note that $\mathbf{y}_{1,\tau_0} = \mathbf{x}$ holds. Therefore $\boldsymbol{\Delta}^{(2\tau_0)} := \mathbf{x} - \mathbf{y}_{2,2\tau_0}$ evolves as

$$\begin{aligned} \dot{\boldsymbol{\Delta}}^{(2\tau_0)} &= \dot{\mathbf{y}}_{1,\tau_0} - \dot{\mathbf{y}}_{2,2\tau_0}, \\ &= \mathbf{g}(\mathbf{y}_{1,\tau_0}, \mathbf{y}_{2,2\tau_0})\boldsymbol{\Delta}^{(2\tau_0)} - \mathbf{k}\boldsymbol{\Delta}^{(2\tau_0)}_{\tau_0}. \end{aligned} \tag{11}$$

This is the same transversal system as Eq. (7), which has a stable fixed point $\boldsymbol{\Delta}^{(\tau_0)} = \mathbf{0}$ by assumption. Therefore, if

T_0 is the transient time needed to sufficiently approach the anticipatory synchronization manifold $\mathbf{x} = \mathbf{y}_{1,\tau_0}$ [which is only a part of the manifold of the augmented system (1,10)], latest after the time mT_0 the last oscillator in the chain will anticipate the driving system with a time $m\tau_0$, and the anticipatory synchronization manifold $\mathbf{x} = \mathbf{y}_{m,m\tau_0}$ will be approached. Choosing m appropriately, in principle, the total anticipation time can be made arbitrarily large.

In practice, although the first part of the augmented system's manifold, $\mathbf{x} = \mathbf{y}_{1,\tau_0}$, may be guaranteed to be stable, the second and subsequent oscillators can happen to become linearly unstable *as long as the system is in a transient motion;* the arguments of $\mathbf{g}$ in Eq. (11) may attain values outside the range of the arguments of $\mathbf{g}$ in Eq. (4) unless the first two oscillators are synchronized. To prevent trajectories to escape to infinity, one can start with only one coupled oscillator and append the next one after time T_0, etc. Here we use an alternative approach: The coupling between the ith and $(i+1)$st oscillator is loosened if the distance between the corresponding states becomes too large by using $\mathbf{K}_i(\mathbf{X}_i - \mathbf{Y}_{i,\tau_0}) = \mathbf{k}(\mathbf{\Delta}_i^{(\tau_0)} \exp[-\frac{(\mathbf{\Delta}_i^{(\tau_0)})^2}{w}])$ in Eq. (10), where the parameter w determines the strength of this correction. (The lower index of $\mathbf{\Delta}$ here denotes the number of the subsystem.)

Numerical simulations for the Rössler and Lorenz oscillators with $m = 12$ (Fig. 2) reveal that the anticipation time can indeed exceed typical time scales of the chaotic oscillations. For much longer runs than shown in Fig. 2, no breakdown of synchronization is observed.

In practical applications where it may not be possible to construct identical copies of the system to be synchronized, the stability of oscillator chains should crucially depend on the system discrepancies. Identical synchronization must break down already for slightly nonidentical systems. In this case, the concept of synchronization still may be useful in an approximate sense, namely, if the trajectories stay close together all the time for small system discrepancies. In this sense, at first sight it may seem that in the anticipatory coupling case the synchronization manifold is much more unstable than in the same chain with conventional coupling. Using coupled maps, we now give a heuristic proof that this intuition can be misleading. The general form considered here is

$$x_{t+1}^{(1)} = \alpha x_t^{(1)} + f(x_{t-1}^{(1)}), \quad x_{t+1}^{(2)} = \alpha x_t^{(2)} + f(x_{t-T}^{(1)}),$$
$$\vdots$$
$$x_{t+1}^{(N)} = \alpha x_t^{(N)} + f(x_{t-T}^{(N-1)}) \qquad (|\alpha| < 1). \tag{12}$$

(Upper indices here denote the number of the chain element.) Here we use complete replacement coupling, with $T = 1$ for identical synchronization and $T = 0$ for the anticipatory case. The oscillator at site i is influenced only by itself and by its left neighbor at site $i - 1$ $(1 < i \leq N)$. In this case of one-dimensional oscillators even global stability results can be derived quite easily [5]. Now we observe the following: Switching from $T = 1$ to $T = 0$, the stability of the whole system is not affected, since in each row of Eq. (12) just the time when the coupling sets in is changed. Therefore, the anticipatory chain is as stable as the chain with identical synchronization, independent of the kind of disturbances imposed onto the systems. Note that the anticipatory complete replacement coupling used in Eq. (12) *locally* is merely a special case of anticipatory dissipative coupling. To see this, rewrite the ith row as $x_{t+1}^{(i)} = \alpha x_t^{(i)} + f(x_{t-1}^{(i)}) + K[f(x_t^{(i-1)}) - f(x_{t-1}^{(i)})]$ $(1 < i \leq N, K = 1)$, expand the coupling term with respect to small state differences and neglect quadratic and higher order terms. Therefore, the conclusion of identical stability properties for the anticipatory and identical synchronization case derived above also may hold locally for this kind of dissipative coupling.

There are crucial differences between identical synchronization in chains of coupled oscillators and synchronization on anticipatory manifolds, which deserve further discussion: (i) A signal can travel against the coupling direction. This in practice causes *seemingly* noncausal behavior, if the deviations from the manifold are so small that they cannot be resolved by measurements anymore. This becomes even more apparent for coupled delayed-feedback systems and coupled map lattices where for certain cases the approach to the anticipatory synchronization manifold can be calculated analytically [4]. For a pair of coupled oscillators, this was recently observed in an electronic circuit [16]. (ii) In technical applications, one would couple an identical system directly to another nonlinear (not necessarily chaotic) system. In contrast to forecasting trajectories by numerical modeling, this way

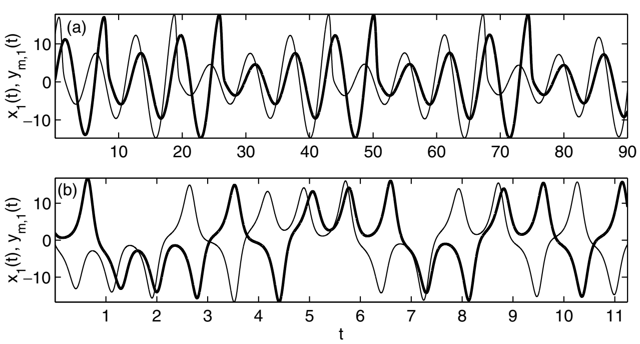

FIG. 2. Numerically simulated time series $x_1(t)$ (bold line) and $y_{m,1}(t)$ with $m = 12$ for the Rössler (a) and Lorenz (b) oscillators, respectively, coupled to the augmented system (10). The resulting anticipation times are $\tau = 12 \times 0.6 = 7.2$ and $\tau = 12 \times 0.075 = 0.9$, respectively, exceeding the typical time scales of the chaotic oscillations. [For the Rössler systems the coupling matrix $\mathbf{k} = \mathrm{diag}(0.8, 0, 0)$ is used; for the Lorenz systems it is $\mathbf{k} = \mathrm{diag}(5, 5, 5)$ [21]. The parameters of the Rössler equations are the same as in Fig. 1; the Lorenz system parameters are $\sigma = 10$, $r = 28$, and $b = 8/3$. In both cases, $w = 10$ is used, and transients are removed.]

forecasting would be possible without any computations necessary. This could be of advantage in fast devices as part of communication systems [17]. (iii) A time lag between the trajectories of coupled systems also can be observed in the phenomenon of *lag synchronization* [18], where two nonidentical systems, usually coupled in a bidirectional manner, approach an approximate bitemporal manifold. Using unidirectional coupling, it is also possible to yield an approximate anticipatory synchronization manifold, but without any memory term involved in the coupling. Since the systems have to be nonidentical, it cannot be expected that the associated augmented system will synchronize in all cases, however. (iv) It can be expected that the maximum anticipation time can be increased using optimized coupling schemes [19].

To summarize, we have shown that the stability of anticipating synchronization can greatly be enhanced by augmenting the phase space of the driven system. This allows for anticipation times that are multiples of the ones that have been accomplished so far using only pairs of synchronizing systems. In particular, it has been shown that the anticipation time may exceed typical time scales of the chaotic dynamics. For the special case of coupled maps with memory, it was shown that the stability of the anticipatory synchronization manifold is the same as that for identical synchronization. This implies that chains of disturbed or slightly nonidentical systems do not behave worse for the anticipatory case, as compared with conventionally coupled chains. Since it is not required that the oscillators are chaotic, we believe that these results pave the way towards real applications. We think it may be worthwhile to look whether nature makes use of these phenomena, e.g., in neuronal information processing.

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*Electronic address: hv@physik.uni-freiburg.de

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