Deterministic Walks in Random Media

Gilson F. Lima,[1,2] Alexandre S. Martinez,[1] and Osame Kinouchi[1]
[1]*Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, Avenida Bandeirantes 3900, CEP 14040-901, Ribeirão Preto, SP, Brazil*
[2]*Escola Técnica Federal de Mato Grosso, R. Zulmira Canavarros 95, CEP 78005-390, Cuiabá, MT, Brazil*
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Deterministic walks over a random set of N points in one and two dimensions ($d = 1, 2$) are considered. Points ("cities") are randomly scattered in $\mathbf{R}^d$ following a uniform distribution. A walker ("tourist"), at each time step, goes to the nearest neighbor city that has not been visited in the past τ steps. Each initial city leads to a different trajectory composed of a transient part and a final p-cycle attractor. Transient times (for $d = 1, 2$) follow an exponential law with a τ-dependent decay time but the density of p cycles can be approximately described by $D(p) \propto p^{-\alpha(\tau)}$. For $\tau \gg 1$ and $\tau/N \ll 1$, the exponent is independent of τ. Some analytical results are given for the $d = 1$ case.

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The study of random walks has been very fruitful in physics and mathematics, and the theory of such stochastic processes is a well-developed subject. The study of deterministic walks is also an interesting subject, but presents the analytical difficulties common to the area of nonlinear dynamical systems and has been less investigated [1]. Here we propose a simple and intriguing problem: a deterministic walk over a random graph with N nodes that is also an example of a local ("on-line") optimization dynamics. It may be called the "local traveling salesman problem" or perhaps the "tourist problem" for short. The deterministic dynamics produces a division of the system phase space in a $\mathcal{O}(N)$ number of attractor basins which trap the walker (ergodicity is broken). The problem is reminiscent of walks in rugged landscapes or zero-temperature spin-glass dynamics but the equivalent of "local minima" are cycles instead of point attractors, as in Kauffman networks [2].

The model is defined as follows: N points are randomly distributed with a uniform density ρ in $\mathbf{R}^d$, where d is the dimensionality of the space. These points may be thought of as "cities" and they may be viewed as vertices of a random graph. At each time step, the "tourist" follows the deterministic rule: *Go to the nearest city that has not been visited in the past τ time steps*. Notice that the tourist wants to minimize only the distance to the next city (a local optimization procedure), not the sum of all distances along the trajectory as in the traveling salesman problem.

Starting from a random city, the tourist performs a trajectory composed of a transient part and a final p-cycle attractor. In this Letter, we report the statistics for some relevant quantities similar to those measured in Kauffman networks [2]: (a) the probability $P_\tau(t)$ for obtaining a transient of size t, defined as the number of steps before the tourist enters some attractor; for large t, it is exponential $P_\tau(t) \propto \exp[-t/\xi(\tau)]$ with the decay time $\xi(\tau)$ growing exponentially for $d = 1$ and linearly for $d = 2$; (b) the density of p cycles $D_\tau(p)$, for $d = 1$, seems to be neither exponential nor a power law, while for $d = 2$, $D_\tau(p)$ follows a power law for nonextreme values $D_\tau(p) \propto p^{-\alpha(\tau)}$ for $\tau \gg 1$, with α independent of τ for $\tau/N \ll 1$; (c) the total density of attractors, $\mathcal{D}(\tau) = \sum_p D_\tau(p)$, is the number of different attractors per city for a given memory, decays exponentially for $d = 1$ and, for $d = 2$, as τ^{-1}; for $d = 1$, $D\xi$ is linear in τ, and for $d = 2$ it is a constant ($\tau \gg 1$ and $\tau/N \ll 1$); and (d) for $d = 1$, the average number $\langle n(p) \rangle$ of cities present in a p cycle which is compared to an analytical result where we show that some cycles are prohibited as odd cycles (apart 3 cycle) and 6 cycle for $\tau = 1$.

Our model can be of general interest for optimization theory with local constraints, nonadditive cost functions, and studies of deterministic dynamical systems with quenched disorder. However, we would like to suggest some more specific motivations for considering this class of problem. The model can be viewed as an example of local foraging strategies [3]. It is arguable that for biological agents (and biologically inspired robots) it could be sometimes more important to minimize the distance traveled in each movement between two safe places instead of optimizing some global cost function [4]. In another spatial scale, cycles could be related to stable migratory routes on environments with localized resources. Local optimization appears due to short range sensorial/cognitive capacities which is a determinant factor in unfamiliar or hostile landscapes [5].

The only nontrivial parameter of the model is the memory window τ. Self-avoidance is limited to this window and trajectories can intersect outside this range. If $\tau = 0$ (no memory) the tourist goes simply to the nearest city until two cities, which are reciprocally nearest neighbors, are found, entering a 2-cycle. For $\tau = N - 2$ the trajectory is totally self-avoiding and one has a kind of traveling salesman problem nearest-neighbor algorithm [4]. The interesting cases are the intermediate ones. For example, if $\tau = 1$, the last visited city cannot be revisited, and only p cycles with $p \geq 3$ can exist. For all τ, the relation $p \geq \tau + 2$ holds.

We stress that this problem is not simply related to geometrical properties since the cycles appear only due to the introduced dynamics. Naively, one could think that a p cycle is a geometrical object, for example a cluster where the distances between the points are smaller than any distance outside the cluster. This indeed is a sufficient but not necessary condition to obtain a p attractor. For example, for $d = 2$ (Fig. 1a), a walker with memory $\tau = 1$ starts from city A and finds the 4-cycle $ABCD$. Although city E is close to the cluster (since $BE < AB$), it is never visited because $BC < BE$ and $CD < CE$. However, if the tourist starts from city C, one gets a 3-cycle that includes city E. This degeneracy and superposition of attractors can be understood noticing that the trajectories of Fig. 1 are represented in configuration space, not in phase space. In phase space, points correspond to $\tau + 1$-tuples $(\mathbf{X}_t, \ldots, \mathbf{X}_{t-\tau})$ where $\mathbf{X}_t$ is the position (x, y) of the tourist at time t and trajectories never intersect. Only for $\tau = 0$ the configuration space is equivalent to the phase space.

In the numerical experiments, N points (with d spatial coordinates) are randomly scattered following a uniform distribution in the interval $[0, 1]^d$. The cities are arbitrarily labeled as $i = 1, \ldots, N$ and one constructs the Euclidean distance matrix $\mathbf{D}$ [6]. Starting from some city, the tourist accesses a transient trajectory until getting trapped in some periodic attractor. The number of steps before the tourist enters the cycle defines the transient length t. The period p and the number of different cities n that pertain to the attractor are also determined. The same city can be visited more than once, thus $n \leq p$.

A finite size study has showed that the behavior of the system is smooth as a function of $1/N$, so we have used $N = 10\,000$ as a reasonable number for our simulations. Periodic boundary conditions have been considered for $d = 1$ and $d = 2$ and do not cause significant differences compared to free boundaries. Since each city is used as a starting point, a landscape with N cities produces N different transients. The statistics over $N_R = 1000$ realizations of cities sets ("maps") are collected. The fluctuation in Figs. 2, 3 and 5 are mainly due to the sampling (N_R), not N. We have $NN_R = 10^7$ trajectories treated for each dimensionality d.

A natural question is about the existence of some critical τ that produces a phase transition (in the thermodynamical limit $N \gg 1$), for example the emergence of an untrapped (percolating) transient state. The distribution of transient times does not suggest this possibility because it is exponential, $P_\tau(t) \propto \exp[-t/\xi(\tau)]$. This is shown in Figs. 2 and 3 for $d = 1$ and $d = 2$, respectively. The memory can be very large ($\tau \gg 1$) but $\tau/N \ll 1$. The exponential decay may be understood as follows. The accumulative of the transient times distribution is the probability of a walker not being trapped in a cycle at t (irrespective to the cycle period p). Consider $q(\tau)$ the probability of the walker not entering a cycle after a movement. The movement of the walker is generated deterministically, but $q(\tau)$ is the same on every movement. The probability of a walker making t steps and not entering a cycle is $P(t) \propto e^{-t/\xi(\tau)}$, where $\xi(\tau) = -1/\ln q(\tau)$. The probability that a walker enters a cycle in a given movement is $\tilde{p} = 1 - q$. Assuming $\tilde{p} \ll 1$, leads to $\tilde{p} = 1/\xi(\tau)$.

For $d = 1$, the characteristic times grow as $\xi(\tau) \propto \exp(\gamma\tau)$ (inset Fig. 2) and the total density decays as $\mathcal{D}(\tau) \propto \exp(-\gamma'\tau)$ (Fig. 4a). For $d = 2$, one observes the linear dependence $\xi(\tau) \propto \tau^\delta$, $\delta = 1.0$ (inset Fig. 3); the total density decays as a power law $\mathcal{D}(\tau) \propto \tau^{-\delta'}$ with $\delta' = 1.0$ (Fig. 4b). The probability $\tilde{p}$ is to a first approximation proportional to $\mathcal{D}(\tau)$. This means that $\mathcal{D}(\tau)\xi(\tau)$ should be constant, that is, $\gamma = \gamma'$ and $\delta = \delta'$. This is indeed the case for large values of τ in $d = 2$ (see Fig. 4c). For $d = 1$, although the exponential terms cancel, a linear factor remains. A better expression for $d = 1$ decay time is $\xi(\tau) = c\tau\exp(\gamma\tau)$. The linear prefactor occurs in both $d = 1$ and $d = 2$, the exponential dependence on τ (in $\mathcal{D}$ Fig. 4a and ξ inset of Fig. 2) is a fingerprint

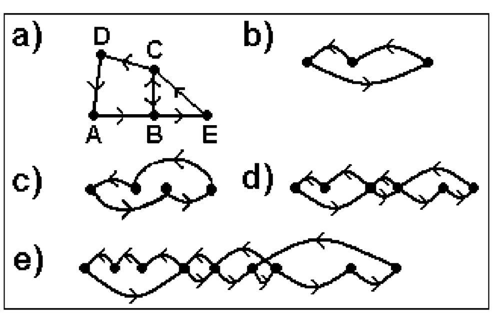

FIG. 1. (a) Example of superposition of attractors for $d = 2$ and $\tau = 1$: starting from A one obtains the 4-cycle $ABCD$, but starting on C one gets the 3-cycle CBE; (b) the 3-cycle base block for $\tau = 1$; (c) the 4-cycle for $\tau = 1$; (d) next permissible cycle for $\tau = 1$: an 8-cycle made of two base blocks (see also Fig. 5a); and (e) an example of odd ($p = 13$) cycle for $\tau = 2$ which is possible because of an internal loop.

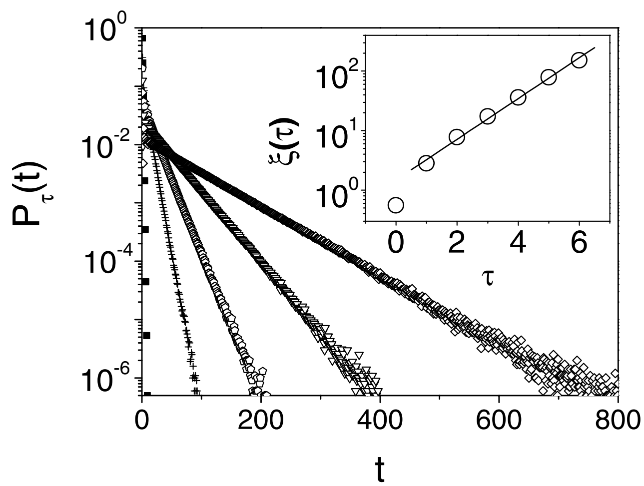

FIG. 2. Distribution of transient times $P_\tau(t)$ for $d = 1$. From left to right: $\tau = 0$, 2, 3, 4, and 5. Inset: Decay time $\xi(\tau)$.

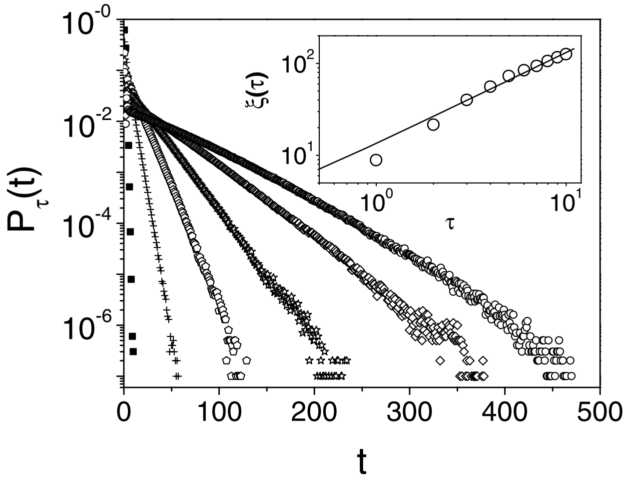

FIG. 3. Distribution $P_\tau(t)$ of transient times for $d = 2$. From left to right: $\tau = 0$, 1, 2, 3, 5, and 10. Inset: Decay time $\xi(\tau)$.

of $d = 1$ and is due to the probability of the occurrence of barriers $[P(d_{i+1} > d_i + d_{i-1} + \ldots + d_{i-\tau})$, where d_i is the distance traveled in step i] which trap the dynamics of a walker into cycles. On the basis of central limit theorem, it is possible to show that this probability decays exponential as a function of τ.

A property of natural interest is the density $D_\tau(p)$ of p cycles, estimated as the number of *different* p cycles divided by N, in the limit of very large systems ($N \gg 1$). Evaluating this quantity requires careful enumeration because, when starting from all the possible initial states, one must not count the same attractor twice. For $d = 1$, some attractors are forbidden as the odd (apart $p = 3$) and $p = 6$ cycles for $\tau = 1$, the presence of long cycles indicates $D_\tau(p)$ is nonexponential, although the evidence for a power law is weak (Fig. 5a). For $d = 2$, one observes, for nonextreme values, a power law $D_\tau(p) \propto p^{-\alpha(\tau)}$ (Fig. 5b) behavior [7–9]. For $\tau \gg 1$ and $\tau/N \ll 1$, the exponent $\alpha = 2.7 \pm 0.2$ is independent of τ (inset Fig. 5b).

The average number of cities $\langle n(p)\rangle$ pertaining to cycles of period p has been studied (Fig. 6). For $d = 1$ there is almost no dispersion in the number of cities per attractor. A p cycle has $n(p)$ cities. We also have found that, for $n > 2(\tau + 2)$, the following relation holds for even cycles: $n(p) = p/2 + \tau + 1$. To see how this relation emerges, notice that for each τ there are configurations of points that constitute barriers to the displacement of the tourist. For example, for $\tau = 1$, a barrier to right displacement appears when $d_{i-1} + d_i < d_{i+1}$. Normally, a tourist will perform a constant direction movement (say, to the right) until a barrier is encountered, then the tourist turns around and moves until another barrier (to left displacements) is found. A cyclic attractor stabilizes, the attractor set being composed by the points between the two barriers.

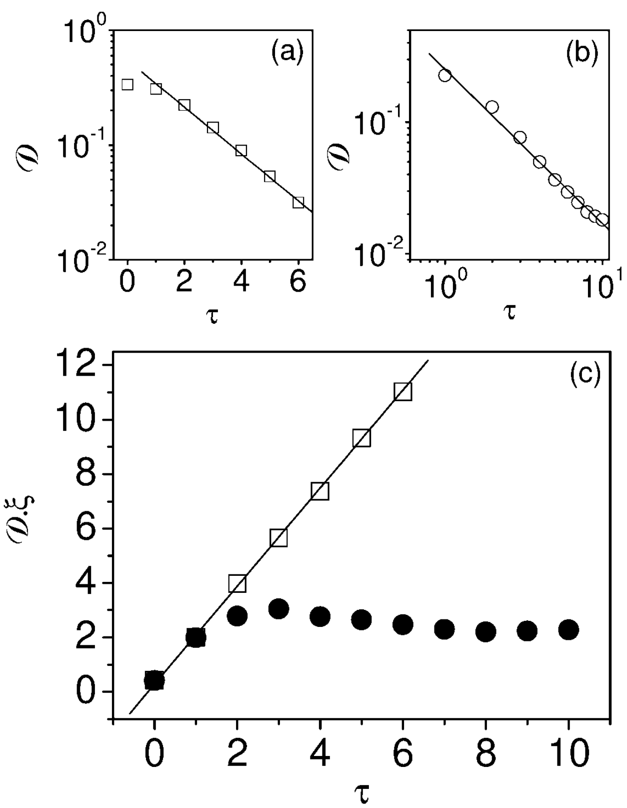

FIG. 4. Total density of attractors $\mathcal{D}(\tau)$: (a) $d = 1$; (b) $d = 2$; and (c) $\mathcal{D}(\tau)\xi(\tau)$ for $d = 1$ (squares) and $d = 2$ (circles), error bars smaller than symbol size.

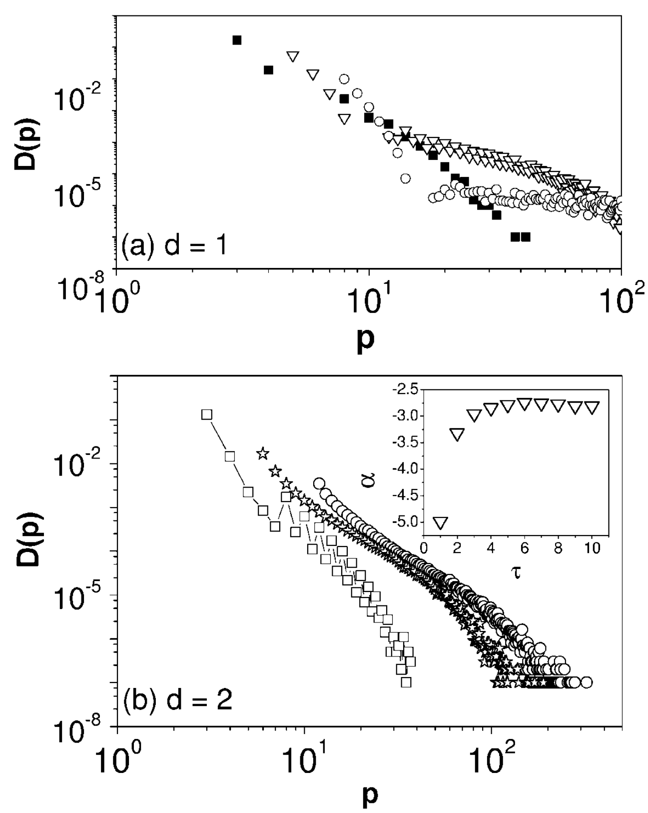

FIG. 5. Examples of attractor densities $\mathcal{D}_\tau(p)$: (a) $d = 1$, $\tau = 1$ (squares), $\tau = 3$ (triangles), and $\tau = 6$ (circles); (b) $d = 2$, $\tau = 1$ (squares), $\tau = 4$ (stars), and $\tau = 10$ (circles), $N = 10\,000$ and $N_R = 1000$. Inset: exponent $\alpha(\tau)$.

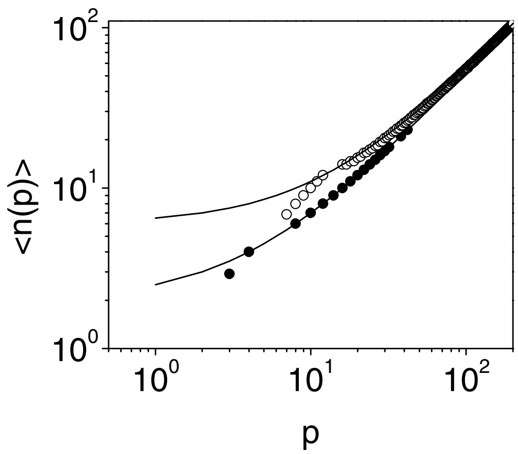

FIG. 6. Number of cities per attractor $\langle n(p) \rangle$. (a) $d = 1$, $\tau = 1$ (filled circles) and $\tau = 6$ (empty circles), theoretical curves (solid) $\langle n(p) \rangle = p/2 + \tau + 1$.

For each τ there is a minimum cycle of period $p_\tau = \tau + 2$, which we call a *base block* (Fig. 1b). A base block is composed of $n_\tau = \tau + 2$ cities. The next cycles follow specific constructions (Fig. 1c). But when n is large, geometrical constraints impose that the most common p cycles are made of two base blocks (one in each attractor extremity) joined by n_I *intermediate cities;* see Fig. 1d. An attractor with n cities thus has $n_I = n - 2n_\tau$ intermediate points. These intermediate cities contribute to the total period with $p_I = 2n_I + 2$ steps (since, for $n_I = 0$, the joining of the base blocks contributes with two steps). Thus, the total period is $p = 2 \times p_\tau + p_I = 2(n - \tau - 1)$, which leads to $n(p) = p/2 + \tau + 1$ (Fig. 6).

This relation holds for cycles with $n \geq 2n_\tau = 2(\tau + 2)$, because only these cycles can incorporate two independent base blocks. For $\tau = 1$, this is the unique conceivable manner of constructing cycles, meaning that odd cycles are prohibited (and also $p = 6$ cycles; see Fig. 5a). For $\tau > 1$, it is possible to construct odd cycles by using internal loops and very specific initial conditions (an example with $\tau = 2$ is given in Fig. 1e).

In $d = 2$, the attractors are polygons with different forms and shapes so that this strict relation $n(p)$ between periods and cities does not hold, although $\langle n \rangle$ also scales linearly with p (not shown). For $\tau = 1$, one finds that odd cycles are less probable than even cycles (Fig. 5b), which is reminiscent of the $d = 1$ behavior. Indeed, this occurs because elongated, quasilinear, odd attractors in two-dimensional space are prohibited by the same geometrical constraints present in the one-dimensional case.

Finally, another analytical result for the $d = 1$ and $\tau = 0$ case can be obtained. It is not hard to see that the distribution of interval sizes follows an exponential distribution $P(d_i) = Ne^{-Nd_i}$ and only 2-cycle attractors exist, corresponding to pairs of reciprocal nearest neighbors. Since the interval sizes are independent variables, one can show that the probability to have mutually nearest neighbors is $P_2 = P(d_{i-1} > d_i \text{ and } d_{i+1} > d_i) = 1/3$, that is, on average, one third of the sequences of four points leads to reciprocal nearest neighbors and so to 2-cycles. Since the number of sequences of four points is, in the large N limit, equal to the number of points, one obtains $D_0(2) = 1/3$. This has been fully confirmed by our numerical simulations [10].

Deterministic walks, which are partially self-avoiding, have been presented in continuous space in one and two dimensions. This simply stated problem presents a rich and highly nontrivial dynamics, where some particular and general aspects have been presented. Also, we have shown that although memory favors space exploitation, it does not lead to an ergodic behavior. Ergodicity can be obtained by a generalization of the proposed model by introducing a stochastic component which will be fully presented and explored in the near future.

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[6] From the **D** matrix another matrix V_{ik} is created where $k = 1, \ldots, \tau + 1$ are the (ranked) nearest neighbors of city i. The dynamics can be performed over the entries of the **V** matrix instead of **D**.

[7] A finite-size study has been performed and is not conclusive since the cutoff lies in the region affected by the statistical fluctuation. A stretched exponential [8] does not fit well large p values. The expression $D(p) \propto p^{-\alpha} e^{-ap}$ with $a = 0.06$ for $\tau = 2$ and $a = 0.01$ for $\tau = 10$ fits well intermediate and high p values. Since the argument of the exponential function is small and a pure power-law fitting, all over the p range, leads to $R^2 > 0.99$ we write simply $D(p) \propto p^{\alpha}$.

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[10] For $\tau = 0$ only 2-cycles exist: $D_0(2) = 0.333 \pm 0.001$ for $d = 1$, and $D_0(2) = 0.31 \pm 0.01$ for $d = 2$.