Dark-Bright Solitons in Inhomogeneous Bose-Einstein Condensates

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We investigate dark-bright vector solitary wave solutions to the coupled nonlinear Schrödinger equations which describe an inhomogeneous two-species Bose-Einstein condensate. While these structures are well known in nonlinear fiber optics, we show that spatial inhomogeneity strongly affects their motion, stability, and interaction, and that current technology suffices for their creation and control in ultracold trapped gases. The effect of controllably different interparticle scattering lengths is examined, and stability against three-dimensional deformations is considered.

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Among the many features of nonlinear equations, the emergence of solitons is one of the most interesting. For the nonlinear Schrödinger equation (NLSE), which governs both nonlinear optical modes in fibers and dilute Bose-Einstein condensates (BECs), two different kinds of scalar solitons, *bright* and *dark,* are known [1]. In optics, spatial bright and dark solitons arise in media with focusing and defocusing nonlinearity, respectively, and for BECs the s-wave scattering interaction is the determining factor (attractive for bright solitons, repulsive for dark). Whereas in gaseous BECs dark solitons only, and only recently, have been observed [2,3], optical solitons are already on the verge of industrial application [4]. In addition to the bright and dark scalar solitons, there are also various multicomponent (vector) solitons known, which arise as solutions to systems of coupled NLSEs. An elegant example is the so-called *dark-bright* soliton, where a bright optical solitary wave exists in a system with *defocusing* nonlinearity because it is trapped within a copropagating dark soliton [5–7]. In this Letter, we investigate the behavior of dark-bright solitons (and solitary waves) in repulsively interacting two-component BECs. We examine the effects of spatial inhomogeneity, three-dimensional geometry, and dissipation, which are all important features of BEC experiments.

In the context of cold atomic gases, the two vector components evolving under the Gross-Pitaevskii NLSE are the macroscopic wave functions of Bose-condensed atoms in two different internal states, which we will denote as $|D\rangle$ and $|B\rangle$. The nonlinear interactions are due to elastic s-wave scattering among the atoms, and are effectively repulsive (positive scattering length) for both systems (^{23}Na and ^{87}Rb) in which multicomponent condensates have been realized [8]. By rescaling lengths, energies, and the wave functions, the general equations may easily be put into the dimensionless form,

$$\begin{aligned} i\dot{\psi}_D &= -\frac{1}{2}\psi_D'' + [V_D + |\psi_D|^2 + g_D|\psi_B|^2 - \mu]\psi_D\,, \\ i\dot{\psi}_B &= -\frac{1}{2}\psi_B'' + [V_B + |\psi_B|^2 + g_B|\psi_D|^2 - \mu - \Delta]\psi_B\,, \end{aligned} \tag{1}$$

where the chemical potentials $\mu_D = \mu$ and $\mu_B \equiv \mu + \Delta$ have been introduced in the usual way. In ^{23}Na and ^{87}Rb, the fortuitous smallness of the triplet scattering length means that the g_j are naturally both close to unity; and quasi-one-dimensional traps that are longitudinally very flat are under active experimental development [9]. So to review the basic properties of dark-bright solitons, we will initially consider the nearly realistic case where $g_j = 1$ and $V_j = 0$, $j = D, B$. The dark-bright soliton solution [5–7] to Eqs. (1) is then given by

$$\psi_D = i\sqrt{\mu}\sin\alpha + \sqrt{\mu}\cos\alpha\tanh\{\kappa[x - q(t)]\}, \qquad \psi_B = \sqrt{\frac{N_B\kappa}{2}}\,e^{i\phi}e^{i\Omega_B t}e^{ix\kappa\tan\alpha}\,\mathrm{sech}\{\kappa[x - q(t)]\}, \tag{2}$$

where $N_B \equiv \int dx|\psi_B|^2$ is the rescaled number of particles in state $|B\rangle$, the soliton inverse length is $\kappa \equiv \sqrt{\mu\cos^2\alpha + (N_B/4)^2} - N_B/4$, the bright component frequency shift is $\Omega_B \equiv \kappa^2(1 - \tan^2\alpha)/2 - \Delta$, and the soliton position is $q(t) = q(0) + t\kappa\tan\alpha$ [see Fig. 1(a)]. The "binding energy" of the bright component in the well formed by the ψ_D mean field is clearly $\kappa^2/2$; the bright component phase shift ϕ is of significance only if there are two or more solitons. Readers familiar with the scalar solitons of the one-component NLSE will recognize ψ_D as a dark (or "grey") soliton of velocity-angle α, and ψ_B as a bright soliton, which can only be found

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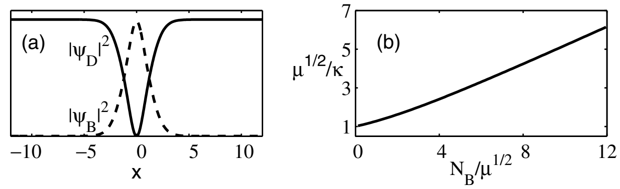

FIG. 1. (a) A dark-bright soliton of Eqs. (2), with $\alpha = 0$. The rescaled densities of $|\psi_B|^2$ and $|\psi_D|^2$ are shown with a broken and a full line, respectively. (b) The size of a motionless dark-bright soliton, in units of the healing length $\mu^{-1/2}$, as a function of $N_B\mu^{-1/2}$.

in single-component condensates if they have negative scattering length. The Thomas-Fermi-like expansion with N_B of the trapped bright component makes the soliton size κ^{-1} longer than for a single-component dark soliton at the same μ [see Fig. 1(b)].

The integrable system of Eqs. (1) with $g_j = 1$ and $V_j = 0$ (known as the Manakov equation [10]) conserves the free energy,

$$G = \frac{1}{2}\int dx[|\psi_D'|^2 + |\psi_B'|^2 + (|\psi_D|^2 + |\psi_B|^2 - \mu)^2 + 2\Delta|\psi_B|^2] = \frac{4}{3}\kappa^3 + \frac{1}{2}N_B\kappa^2(1 + \tan^2\alpha) + N_B\Delta . \quad (3)$$

Since G decreases with increasing soliton velocity, the soliton is formally unstable (to acceleration). But one implication of integrability is that perturbations of Eqs. (2) due to interactions with other waves (solitary or ordinary) will not cause dissipation. If an inhomogeneous potential is added, however, by allowing nonzero V_j in (1), then the system is no longer integrable, and the soliton can interact nontrivially with the surrounding condensate. Nevertheless, if spatially V_j vary slowly on the soliton scale κ, then in the frame comoving with the soliton the potentials are slowly varying in time. As in the case of the scalar dark soliton [11], a multiple time scale boundary layer analysis shows that the soliton energy $G(q, N_B, \alpha)$, given by replacing $\mu \rightarrow mu - V_D(q)$ and $\Delta \rightarrow \Delta - V_B(q) + V_D(q)$ in Eq. (3), is an adiabatic invariant [12]. The constancy of G, up to second order in the ratio of soliton and potential length scales, determines to the same accuracy the motion of the soliton in the potential.

This motion simplifies for soliton speeds $|\dot{q}| = |\kappa \tan\alpha| \ll 1$ (the speed of sound), because

$$G = \frac{4}{3}\left[\mu + \frac{N_B^2}{16} - V_D(q)\right]^{3/2} + N_B\left[V_B(q) - \frac{V_D(q)}{2}\right] - 2\dot{q}^2\sqrt{\mu + (N_B/4)^2 - V_D(q)} + \mathcal{O}(\dot{q}^4), \quad (4)$$

dropping a term which is adiabatically constant. This implies the low-velocity equation of motion,

$$\ddot{q} = -\frac{V_D'(q)}{2} - \frac{N_B[V_D'(q) - 2V_B'(q)]}{8\sqrt{\mu + (N_B/4)^2 - V_D(q)}}, \quad (5)$$

which, together with its numerical confirmation shown in Fig. 2, is the primary result reported in this paper. In the limit $N_B \rightarrow 0$, we recover the equation of motion of the dark soliton [11], and as N_B increases we find that the soliton is more and more insulated from the effect of V_D, and more sensitive to $V_B - V_D$. In the limit $N_B \gg \sqrt{\mu}$, where the soliton is expanded by the large bright component to many healing lengths in size, we have

$$\ddot{q} = \left(1 - 8\frac{\mu - V_D}{N_B^2}\right)[V_B'(q) - V_D'(q)] - 4\frac{\mu - V_D}{N_B^2}V_D'(q), \quad (6)$$

so that a small differential force on the bright component will predominate. Our assumption that the whole soliton is small compared to the trap scale, however, means that the dark component retains its dramatic effect of giving the soliton an effectively negative mass: the soliton accelerates in the opposite direction to a force exerted through V_B. If V_B and V_D are equal, on the other hand, a highly expanded dark-bright soliton with $N_B \gg \sqrt{\mu}$ moves in the potential as if it had a very large positive mass (because, as one can see from Eq. (4), the soliton's potential energy is also $\sim -V_D$). Numerical integration of the coupled NLSEs shows excellent agreement with Eq. (5) (see Fig. 2). Note that, for harmonic $V_B > V_D$, Eq. (5) implies that there is a critical N_B above which the soliton will be driven out of the trap instead of oscillating. (Scalar dark or bright solitons subjected to such potentials would both oscillate nicely.) While the precise transition point between very slow oscillation and very slow escape is difficult to check

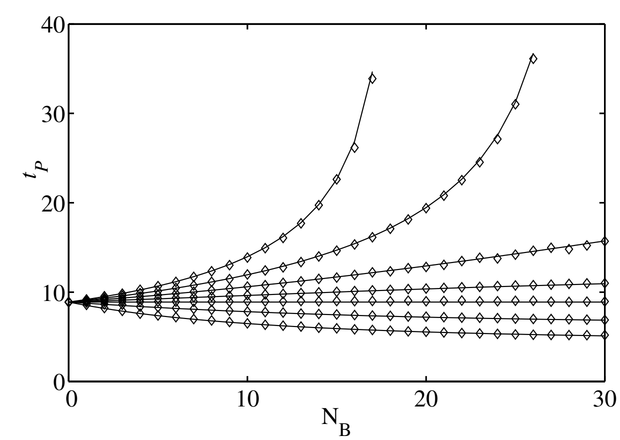

FIG. 2. Period of oscillation for a dark-bright soliton in a harmonic trap calculated numerically (diamonds) and from Eq. (5) (solid line). The potentials are related as $V_B = \gamma V_D$, with $\gamma = -1, -0.5, 0.5, 0.75, 1, 1.25$, and 1.5, respectively, for the graphs starting from below. In all cases $\int dx|\psi_D|^2 = 1000$, so that μ ranges from 66.2 to 67.1. The divergence of the curves for $\gamma = 1.25$ and $\gamma = 1.5$ shows the breakdown of the oscillations.

numerically, our simulations confirm that escape does occur.

A trapping potential also modifies the effects of the short-ranged interactions between solitons. Although the respective bright component numbers $\propto N_B$ of two solitons are simply conserved during such interactions, the relative phase of the two bright components strongly affects the details of the interaction [13,14]: dark-bright solitons repel each other when the phase difference between the bright components is $\Delta\phi = \phi_1 - \phi_2 = 0$, and attract each other when this difference is $\Delta\phi = \pi$. This short range behavior, which is opposite to that of scalar bright solitons, occurs independently of the confining potential, but if the effect of a potential is to keep two solitons close together, then their phase-dependent interaction can significantly affect their oscillations: see Fig. 3. Moreover, even if the trap does not confine both solitons within their interaction range, the inhomogeneous potential will still qualitatively modify the effects of soliton collisions. Without a potential, the only net effect of a collision is a "jump-like" spatial translation of the solitons, relative to where each would have been if it had not encountered the other. (See [15] for a detailed study of dark-bright soliton collisions at $V_j = 0$.) Figure 4 shows that, in a trap, such a translation can transfer energy between solitons.

In addition to the inhomogeneous potential, integrability may also be destroyed by the fact that the off-diagonal interaction strengths g_j will generally differ from unity. In the quasi-1D limit, a condensate has three independent interaction coefficients $\gamma_{jk} = 2a_{jk}/(A_j + A_k)$, where $a_{jk} = a_{kj}$ is the 3D s-wave scattering length for collisions between atoms in states j and k, and A_j is the cross-sectional area of the trap confining species $(j, k = D, B)$. We then rescale $\psi_{D,B}$ to set the diagonal coefficients to unity $(\psi_j \to \psi_j/\sqrt{\gamma_{jj}})$, obtaining the two off-diagonal coefficients $g_j \equiv \gamma_{DB}/\gamma_{jj}$ which appear in (1). One can therefore show that

$$\begin{aligned} g_D &= \frac{a_{DB}}{a_{DD}}\left(1 + \frac{A_D - A_B}{A_D + A_B}\right), \\ g_B &= \frac{a_{DB}}{a_{BB}}\left(1 - \frac{A_D - A_B}{A_D + A_B}\right), \end{aligned} \tag{7}$$

so that varying the relative tightness of radial confinement for the two species yields one free control parameter, which can allow retuning of g_j even without the measure of modifying the scattering lengths themselves.

If $g_j \neq 1$, solutions that are distorted versions of the dark-bright soliton certainly exist [15,16], although they may only be given in closed form for special cases. (For instance, in the limit of small N_B, one may discard the $|\psi_B|^2$ terms in the NLSE, and find dark soliton solutions for ψ_D, with $\psi_B \propto \mathrm{sech}^\nu[\kappa(x - q)]$ for $\nu(\nu + 1) = 2g_D$. For $g_D > 1$, there are also one or more excited bound states of ψ_B.) Such solutions are often referred to as solitary waves rather than solitons, to indicate that they may interact nontrivially with other solitary or ordinary waves. This means that a collision between dark-bright solitary waves may cause a net transfer of a bright component from one solitary wave to the other; see Fig. 5. It also implies that, unlike true solitons, which are transparent to all quasiparticle modes, dark-bright solitary waves will suffer from dissipation due to collisions with thermal particles and phonons, even when the one-dimensional approximation is excellent. It is a problem beyond the scope of this Letter to compute scattering rates with g_j significantly different from unity. For g_j close to unity, however, simple estimates for the antidamping rate show it to be negligible, at attainably low temperatures, because the cross sections for dissipative collisions are proportional to the squares of the scattering length differences.

In current experiments, however, soliton lifetimes are limited not by one-dimensional dissipation, but by the breakdown of the one-dimensional approximation and the onset of transverse dynamical instability. As shown by Muryshev *et al.* [17], a single-component dark soliton is unstable to transverse excitations of wavelength greater than the soliton size, so that radial confinement to within a healing length should stabilize dark solitons. Extending their method of analysis [12], one can show that the dark-bright soliton is also stable against transverse instabilities of wavelength less than its size κ^{-1}. Since for $N_B \gg \sqrt{\mu}$ this can be much larger than the healing length, dark-bright solitons should be more stable than pure dark solitons, even in traps that do not attain the quasi-1D regime.

Even more conveniently, we note that a robust method for the controlled creation of dark-bright solitons has already been presented and analyzed in detail (without being explicitly recognized as such) [18]. Dum *et al.* have shown

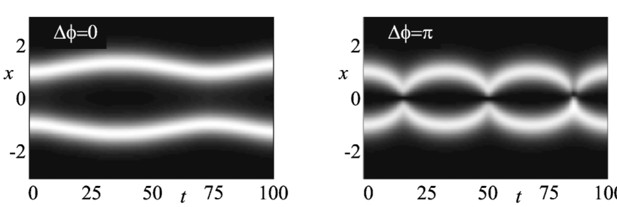

FIG. 3. Symmetric collision of two dark-bright solitons in a harmonic trap ($\gamma = 1$). Degree of brightness indicates $|\psi_B|^2$ as a function of x and t, for repulsive ($\Delta\phi = 0$) and attractive ($\Delta\phi = \pi$) interaction.

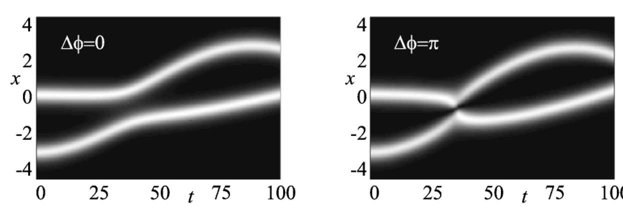

FIG. 4. Collision of two dark-bright solitons in a harmonic trap ($\gamma = 1$). Shown is $|\psi_B|^2$ as a function of x and t. Initially one soliton is at rest (i.e., $q = 0$); after the collision both solitons are oscillating.

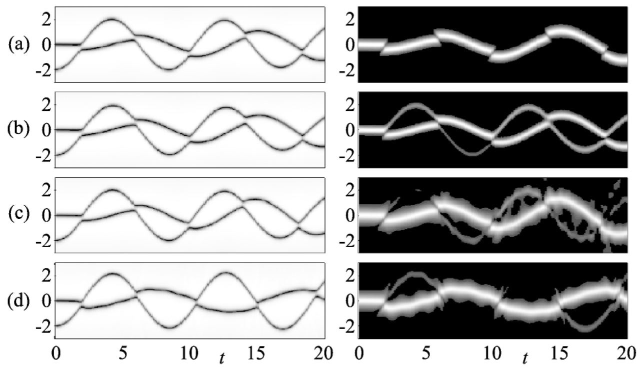

FIG. 5. Collisions between an initially dark soliton ($N_B = 0$) and a dark-bright solitary wave initially at rest in a harmonic trap. Each row is a separate evolution, with the left and right plots showing $|\psi_D|^2$ and $|\psi_B|^2$, respectively. In all cases $\int dx|\psi_D|^2 = 400$ and $\int dx|\psi_B|^2 = 4$, for $\mu \doteq 36$. From top to bottom, we have (g_D, g_B) as follows: (a) (1.03,1.03); this case is not distinguishable from (1,1). (b) (1.5,1.5); transfer of bright component occurs. (c) (0.5,0.5); scattering as well as transfer of bright component is seen in the last collision. (d) (0.5,1.5); viewing a true "movie" of $|\psi_D|^2$ reveals that, during the collisions, the background cloud is much more significantly excited in this case than in the others, and this is the reason for the noticeably different soliton motion in this case. The case (1.5,0.5), which is not shown here, is just noticeably different from (1.5,1.5).

that dark solitons may be created in one condensate component by adiabatic transfer of population from a condensate in another internal state, and have already noted that in the late stages of this procedure the second component appears as a stretched dark soliton around the remaining population of the first, whose wave function approaches a hyperbolic secant. Stopping short of complete adiabatic transfer will, in fact, produce a dark-bright soliton of arbitrary N_B. Masked Rabi transfer with phase imprinting may also be possible, and the smoother total density profile and larger size of a highly expanded dark-bright soliton should simplify the creation of slow and stable solitons by this method [19].

Our conclusion is that dark-bright solitons move in trapped condensates much as dark solitons (though more slowly, if $V_D = V_B$), and have strong advantages in stability and controllability. Since $|\psi_B|^2$ can be imaged separately and nondestructively, they also offer a realization of Reinhardt's and Clark's proposal to track solitons by trapping distinguishable atoms inside them [20]. And, in addition to advancing soliton studies into the inhomogeneous regime of BECs, production of dark-bright solitons in BECs would be the development of atom optical tweezers with potentially submicron precision: the trapping and manipulation of ultracold atoms by ultracold atoms.

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[1] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972); **37**, 823 (1973).
[2] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. **83**, 5198 (1999).
[3] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Philips, Science **287**, 97 (2000).
[4] Y. S. Kivshar and B. Luther-Davies, Phys. Rep. **298**, 81 (1998); H. A. Haus and W. S. Wong, Rev. Mod. Phys. **68**, 423 (1996).
[5] S. Trillo, S. Wabnitz, E. M. Wright, and G. I. Stegeman, Opt. Lett. **13**, 871 (1988).
[6] D. N. Christodoulides, Phys. Lett. A **132**, 451 (1988).
[7] M. Shalaby and A. J. Barthelemy, IEEE J. Quantum Electron. **28**, 2736 (1992).
[8] For a detailed introduction, see *Bose-Einstein Condensation in Atomic Gases,* edited by M. Inguscio, S. Stringari, and C. Wieman (IOP Press, Amsterdam, 1999).
[9] K. Bongs, S. Burger, S. Dettmer, D. Hellweg, J. Arlt, W. Ertmer, and K. Sengstock, Phys. Rev. A **63**, 031602(R) (2001); W. Ketterle (private communication).
[10] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[11] Th. Busch and J. R. Anglin, Phys. Rev. Lett. **84**, 2298 (2000).
[12] Th. Busch and J. R. Anglin (to be published).
[13] E. A. Ostrovskaya, Y. S. Kivshar, Z. Chen, and M. Segev, Opt. Lett. **24**, 327 (1999).
[14] E. A. Ostrovskaya and Y. S. Kivshar, Opt. Lett. **23**, 1268 (1998).
[15] A. P. Sheppard and Y. S. Kivshar, Phys. Rev. E **55**, 4773 (1997).
[16] F. T. Hioe, Phys. Rev. Lett. **82**, 1152 (1999).
[17] A. E. Muryshev, H. B. van Linden van den Heuvell, G. V. Shlyapnikov, Phys. Rev. A **60**, R2665 (1999).
[18] R. Dum, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. **80**, 2972 (1998).
[19] Shortly after submission of this Letter, a paper appeared in which experimental creation of a dark-bright soliton state is reported: B. P. Anderson *et al.,* cond-mat/0012444, 2000.
[20] W. P. Reinhardt and C. W. Clark, J. Phys. B **30**, L785 (1997).