Critical Exponents of the Gross-Neveu Model from the Effective Average Action

L. Rosa, P. Vitale,* and C. Wetterich

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120, Heidelberg, Germany

(Received 19 July 2000)

The phase transition of the Gross-Neveu model with N fermions is investigated by means of a nonperturbative evolution equation for the scale dependence of the effective average action. The critical exponents and scaling amplitudes are calculated for various values of N in $d = 3$. It is also explicitly verified that the Neveu-Yukawa model belongs to the same universality class as the Gross-Neveu model.

DOI: 10.1103/PhysRevLett.86.958 PACS numbers: 11.10.Kk, 05.90.+m, 11.10.Hi, 11.30.Qc

The Gross-Neveu (GN) model [1] is one of the simplest models for interacting fermions. Nevertheless, in three dimensions our quantitative understanding beyond some universal characteristics of the phase transition has remained rather incomplete. The universality class of the GN model in dimensions between 2 and 4 has been argued to be the same as the Neveu-Yukawa (NY) model [2] in $4 - \epsilon$ dimensions [3]. Both the large N and ϵ expansion indicate that a second order phase transition takes place for some critical value of the coupling constant if the number of fermion species N is larger than one [4]. The anomalous dimensions have been calculated up to the third order in the $1/N$ expansion [5], while some critical exponents have been computed to the order $1/N$ in the phase with spontaneous symmetry breaking (SSB) [6]. In this Letter we find the second order phase transition and calculate the critical exponents employing an analytical method based on nonperturbative flow equations for scale dependent effective couplings. We directly obtain results for arbitrary dimension and without a restriction to large N. Despite the presence of massless fermions we are able to investigate the symmetric phase. Because of the fermion fluctuations the infrared physics is not trivial in the NY language and requires a careful discussion of the critical exponents. Beyond the universal critical behavior our method gives a description for arbitrary values of the GN coupling away from the critical point. In particular, we compute the nonuniversal critical amplitudes.

The running couplings parametrize the effective average action Γ_k [7] which is a type of coarse grained free energy. It includes the effects of the quantum fluctuations with momenta larger than an infrared cutoff k. In the limit where the average scale k tends to zero Γ_k becomes therefore the usual effective action, i.e., the generating functional of $1PI$ Green functions. In the limit $k \to \infty$ it approaches the classical action. In a theory with bosons and fermions the scale dependence of Γ_k can be described by an exact nonperturbative evolution equation [7,8]

$$\frac{\partial}{\partial t}\Gamma_k[\phi,\psi] = \frac{1}{2}\mathrm{Tr}\left\{(\Gamma_k^{(2)} + \mathcal{R}_k)_B^{-1}\frac{\partial}{\partial t}R_{kB} - (\Gamma_k^{(2)} + \mathcal{R}_k)_F^{-1}\frac{\partial}{\partial t}R_{kF}\right\}, \tag{1}$$

where $t = \ln(k/\Lambda)$ with Λ some suitable high momentum scale. The trace represents a momentum integration as well as a summation over internal indices and $\Gamma_k^{(2)}$ is the exact inverse propagator given by the matrix of second functional derivatives of the action with respect to bosonic and fermionic field variables. We parametrize the infrared cutoff

$$\mathcal{R}_k(q,q') = \begin{pmatrix} R_{kB} & 0 & 0 \\ 0 & 0 & R_{kF} \\ 0 & -R_{kF} & 0 \end{pmatrix}(2\pi)^d\delta^d(q - q')$$

by means of the bosonic and fermionic cutoff functions $R_{kB}(q) = q^2 Z_{\sigma,k} r_B(q)$, $R_{kF}(q) = i\not{q} Z_{\psi,k} r_F(q)$. We choose

$$R_{kB} = \frac{Z_{\sigma,k}q^2}{e^{q^2/k^2} - 1}; \qquad R_{kF} = iZ_{\psi,k}\not{q}\left(\frac{1}{\sqrt{1 - e^{-\frac{q^2}{k^2}}}} - 1\right),$$

where $Z_{\sigma,k}, Z_{\psi,k}$ are wave function renormalizations. The momentum integration in Eq. (1) is both infrared and ultraviolet finite. Equation (1) is an exact but complicated functional differential equation which can be solved only approximately by truncating the most general form of Γ_k. Once a suitable nonperturbative truncation is found the flow equation can be integrated from some short distance scale Λ, where Γ_Λ can be taken as the classical action, to $k \to 0$ thus solving the model approximately.

The GN model is described in terms of a $O(N)$ symmetric action for a set of N massless Dirac fermions. The classical Euclidean action is given by

$$S_{GN} = \int d^d x\left\{\frac{1}{2\bar{G}}\sigma^2(x) - \bar{\psi}_i(x)[\not{\nabla} + \sigma(x)]\psi_i(x)\right\}.$$

(Here and in the following we distinguish with a bar the dimensionful couplings.) The (pseudo)-scalar $\sigma(x)$ is an auxiliary nondynamical field which can be integrated out from the partition function, leading to the replacement $\sigma(x) \to \bar{G}\bar{\psi}(x)\psi(x)$. Its vacuum expectation value σ_0 is proportional to the fermion condensate $\sigma_0 = \bar{G}\langle\bar{\psi}\psi\rangle$. The model is asymptotically free and perturbatively renormalizable in two dimensions, hence it exhibits a nontrivial fixed point in $d = 2 + \epsilon$. It is $1/N$ renormalizable in $2 < d < 4$.

 0031-9007/01/86(6)/958(4)$15.00

The NY model whose classical action is

$$S_{NY} = \int d^d x \left\{ - \bar{\psi}_i(x)[\not\nabla + \bar{h}\sigma(x)]\psi_i(x) + \frac{1}{2}[\partial_\mu \sigma(x)]^2 + \frac{m^2}{2}\sigma^2(x) + \frac{\bar{g}}{4!}\sigma^4(x) \right\}$$

has a Gaussian fixed point in $d = 4$ where it is perturbatively renormalizable and a nontrivial fixed point in $d = 4 - \epsilon$. Both models have, in even dimensions, a discrete chiral symmetry which prevents the addition of a fermion mass term, while in odd dimensions a mass term is forbidden by space parity. Performing a large N analysis, the universal properties of the two models are argued to be the same in $2 < d < 4$ [3]; in such limit the two models are equivalent in the scaling region if we rescale $\bar{h}\sigma$ to σ and set $\bar{G} = \bar{h}^2/m^2$.

We consider a truncation of the effective action Γ_k which contains a potential for the scalar field and a Yukawa term. In momentum space it reads [$\int dq = \int d^d q/(2\pi)^d$]

$$\Gamma_k[\sigma, \psi, \bar{\psi}] = \int d^d x\, U_k(\sigma) + \int dq \Bigg[\frac{Z_{\sigma,k}}{2}\sigma(-q) q^2 \sigma(q) - Z_{\psi,k}\bar{\psi}_i(-q) i\not{q} \bar{\psi}^i(q) - \int dp\, \bar{h}_k \bar{\psi}_i(-q)\sigma(p)\sigma(p)\psi^i(q-p) \Bigg]. \tag{2}$$

The scalar potential is assumed to be a function of the invariant $\sigma^2(x)$, and we make the further simplification

$$U_k(\sigma) = \frac{m_k^2}{2}[\sigma^2(x) - \sigma_{0k}^2] + \frac{\bar{g}_k}{4!}[\sigma^2(x) - \sigma_{0k}^2]^2 + \frac{\bar{b}_k}{6!}[\sigma^2(x) - \sigma_{0k}^2]^3. \tag{3}$$

The symmetric regime is characterized by the minimum being at $\sigma_{0k}^2 = 0$. In the SSB regime a k-dependent minimum $\sigma_{0k}^2 \neq 0$ develops, whereas $m_k^2 = 0$.

Inserting Eqs. (2) and (3) into (1), we obtain a set of evolution or renormalization group equations (RGE) for the effective parameters of the theory in the two regimes. We find it convenient to work with dimensionless quantities $h_k^2 = Z_\sigma^{-1} Z_\psi^{-2} k^{d-4}\bar{h}_k^2$, $g_k = Z_\sigma^{-2} k^{d-4}\bar{g}_k$, $b_k = Z_\sigma^{-3} k^{2d-6}\bar{b}_k$, $e_k = Z_\sigma^{-1} k^{-2} m_k^2$, $\tilde{\rho} = \frac{1}{2} Z_\sigma k^{2-d}\sigma^2$, $\kappa_k = \frac{1}{2} Z_\sigma k^{2-d}\sigma_{0k}^2$, $u_k = U_k k^{-d}$, and we use $u_k' = \frac{\partial u_k}{\partial \tilde{\rho}}$, etc. The evolution equation for the potential obtains from (1) by evaluating $\Gamma_k^{(2)}$ in the truncation (2) for a constant background scalar field. We find

$$\frac{\partial_t U_k(\sigma)}{k^d} = v_d \int_0^\infty dy\, y^{d/2} \left\{ \frac{-\eta_\sigma r_B - 2y\dot{r}_B}{u_k' + 2\tilde{\rho} u_k'' + y(1 + r_B)} + 2N' \frac{(\eta_\psi r_F + 2y\dot{r}_F)(1 + r_F)}{2h_k^2\tilde{\rho} + y(1 + r_F)^2} \right\}. \tag{4}$$

Here we have introduced the notation $N' = 2^{\gamma/2}N$ with $2^{\gamma/2}$ the dimension of the γ matrices and $y = q^2/k^2$, $\dot{r} = \frac{\partial r}{\partial y}$, and $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$. The anomalous dimensions η_σ and η_ψ are defined as

$$\eta_\sigma(k) = -\partial_t \ln Z_{\sigma,k}, \qquad \eta_\psi(k) = -\partial_t \ln Z_{\psi,k}, \tag{5}$$

where the wave function renormalizations Z parametrize the momentum dependence of the propagators at zero momentum and $\sigma = \sigma_{0k}$. One finds

$$\eta_\sigma(k) = \partial_\alpha \Bigg[\left(\frac{2}{15} b_k \kappa_k + g_k\right)^2 \kappa_k \frac{v_d}{d} \int dy\, y^{d/2} \times \{2N' h_k^2 [2h_k^2 \kappa_\kappa \dot{F}(y,\alpha)^2 - y\dot{G}(y,\alpha)^2] + \dot{H}(y,\alpha)^2\} \Bigg]_{\alpha=0}, \tag{6}$$

$$\eta_\psi(k) = 4h_k^2 \partial_\alpha \left\{ \frac{v_d}{d} \int dy\, y^{d/2} \dot{H}(y,\alpha) G(y,\alpha) \right\}_{\alpha=0}, \tag{7}$$

with

$$H(y,\alpha) = \frac{1}{[e_k + \frac{2}{3} g_k \kappa_k + y(1 + r_B) - \alpha(\eta_\sigma r_B + 2y\dot{r}_B)]},$$

$$F(y,\alpha) = \frac{1}{y[1 + r_F - \alpha(\eta_\psi r_F + 2y\dot{r}_F)]^2 + 2h_k^2 \kappa_k},$$

$$G(y,\alpha) = F(y,\alpha)[1 + r_F - \alpha(\eta_\psi r_F + 2y\dot{r}_F)].$$

Finally, the evolution equation for the Yukawa coupling obtains from taking derivatives of Eq. (1) with respect to $\bar{\psi}$, ψ, and σ:

$$\partial_t h_k^2 = (2\eta_\psi + \eta_\sigma + d - 4) - 4h_k^4 v_d \int_0^\infty dy\, y^{d/2} \left[(\eta_\sigma r_B + 2y\dot{r}_B) H^2(y,0) F(y,0) - 2(\eta_\psi r_F + 2y\dot{r}_F) \frac{G^2(y,0) H(y,0)}{1 + r_F} \right]. \tag{8}$$

Equations (4)–(8) are valid in both regimes provided we set κ_k and e_k appropriately.

If we expand this set of equations in the coupling constants we recover the one-loop results obtained in the $4 - \epsilon$ expansion for the NY model [2]

$$\partial_t g = -\epsilon g + \frac{1}{8\pi^2}\left(\frac{3}{2} g^2 + 4Ngh^2 - 24Nh^4\right),$$

$$\partial_t h^2 = -\epsilon h^2 + \frac{h^4}{8\pi^2}(2N + 3),$$

$$\eta_\sigma = \frac{Nh^2}{4\pi^2}, \quad \eta_\psi = \frac{Nh^2}{16\pi^2}.$$

Moreover, after identifying the running coupling constant of the GN model as $G = h_k^2/e_k$ we also recover the one-loop result obtained in the $2 + \epsilon$ expansion for the GN model [2]

$$\partial_t G = (d - 2)G - (N' - 2)\frac{G^2}{2\pi} + O(G^3). \quad (9)$$

We numerically evolve the flow equations (4)–(8) from a large momentum scale Λ to $k \to 0$. The initial values of the parameters are chosen in such a way that $\Gamma_\Lambda = S_{GN}$: $Z_{\sigma\Lambda} = 10^{-10}$, $Z_{\psi\Lambda} = 1$, $\bar{h}_\Lambda^2 = \Lambda$, $\bar{g}_\Lambda = 0$, $\bar{b}_\Lambda = 0$. Then $e_\Lambda = (Z_{\sigma\Lambda}G_\Lambda)^{-1}$ is the only free parameter of the theory and plays the role of the temperature. It has to be tuned in order to be near the second order phase transition. For the value $e_{\Lambda cr}$ corresponding to the critical temperature T_c the flow ends in a fixed point with constant $e_k > 0$. The relevant parameter for the deviation form T_c is $\delta e = e_\Lambda - e_{\Lambda cr} = H(T - T_c)$ with constant H.

In the symmetric (high T) phase the fermions are massless. Their fluctuations induce a nontrivial dependence of Z_σ and the renormalized scalar mass $m_R^2(k) = Z_\sigma^{-1}(k)m^2(k)$ on the scale k even away from the phase transition. This contrasts with the standard situation where the running of $m_R(k)$ essentially stops in the symmetric phase once k becomes much smaller than m_R. The issue of critical exponents in a situation with two different infrared cutoffs k and m_R is therefore more complex than usual. In a standard situation we would define the exponents γ and ν by following the temperature dependence of the unrenormalized and renormalized mass $m^2(k)$ and $m_R^2(k)$ for $k \to 0$ [9]. Here we define the renormalized mass at some fixed small ratio k/m_R by

$$\bar{m}_R^2 = m_R^2(k_c) - m_{Rcr}^2(k_c), \qquad k_c = r_c\bar{m}_R, \quad (10)$$

with $m_{Rcr}^2(k) = ek^2$ on the critical trajectory. (In the numerical simulations we fix the ratio r_c to be equal to 0.01.) This mass corresponds to the only relevant parameter characterizing the critical behavior. It is directly related to the deviation from the critical temperature δe. We also define the inverse susceptibility or unrenormalized mass by

$$\bar{m}^2 = \bar{m}_R^2 Z_\sigma(k_c, \bar{m}_R). \quad (11)$$

Correspondingly, the critical exponents ν and γ are defined for fixed r_c

$$\nu = \frac{1}{2}\lim_{\delta e\to 0}\frac{\partial \ln\bar{m}_R^2(\delta e)}{\partial \ln\delta e}, \qquad \gamma = \lim_{\delta e\to 0}\frac{\partial \ln\bar{m}^2(\delta e)}{\partial \ln\delta e}. \quad (12)$$

From the definition (11) one has the relation

$$2\nu = \gamma - \frac{\partial \ln Z_\sigma(k_c, \bar{m}_R)}{\partial \ln\delta e}. \quad (13)$$

A typical form of Z_σ is

$$Z_\sigma \simeq Z_0\left(\frac{\bar{m}_R^2 + k^2}{\Lambda}\right)^{-(1/2)\bar{\eta}_\sigma}\left(\frac{k^2}{\bar{m}_R^2 + k^2}\right)^{-(1/2)\eta_2},$$

and we conclude

$$\frac{\partial \ln Z_\sigma(k_c, \bar{m}_R)}{\partial \ln\delta e} = -\bar{\eta}_\sigma\nu, \qquad \gamma = \nu(2 - \bar{\eta}_\sigma). \quad (14)$$

This is the usual index relation. The index η_2, which vanishes in the standard situation, determines the dependence of $\bar{m}_{cr}^2(k, \delta e) = m_{Rcr}^2(k)Z_\sigma(k, \delta e)$. on r_c. In Table I we summarize the results obtained for $N = 2, 3, 4, 12$ in $d = 3$ dimensions.

For a more detailed understanding of the scale dependence we consider next the running of the renormalized mass and unrenormalized mass with k. We define

$$\hat{\nu}(k, \delta e) = \frac{1}{2}\frac{\partial \ln[m_R^2(k, \delta e) - m_{Rcr}^2(k)]}{\partial t}\bigg|_{\delta e}, \quad (15)$$

$$\hat{\gamma}(k, \delta e) = \frac{\partial \ln[m^2(k, \delta e) - \bar{m}_{cr}^2(k, \delta e)]}{\partial t}\bigg|_{\delta e}, \quad (16)$$

with $\bar{m}_{cr}^2(k, \delta e) = m_{Rcr}^2(k)Z_\sigma(k, \delta e)$. The relation $m_R^2 = m^2/Z_\sigma$ implies the index relation

$$\hat{\gamma} = 2\hat{\nu} - \eta_\sigma, \quad (17)$$

which differs from the usual relation $\gamma = \nu(2 - \eta_\sigma)$. For both k and $\bar{m}_R$ sufficiently small and $k \gg \bar{m}_R$ the indices $\hat{\nu}$, $\hat{\gamma}$, η_σ, and η_ψ approach constant values independent of k and $\bar{m}_R$. We fix $d = 3, N = 3$, and

TABLE I. Critical exponents and amplitudes for different values of N, $d = 3$.

N	2	3	4	12
ν	0.961	1.041	1.010	1.023
γ	1.384	1.323	1.228	1.075
$\nu(2 - \bar{\eta}_\sigma)$	1.403	1.323	1.230	1.075
β	0.745	0.903	0.910	0.998
$\frac{\nu}{2}(1 + \eta_\sigma)$	0.750	0.890	0.903	0.991
A_ν/Λ	0.007	0.016	0.009	0.014
A_γ/Λ^2	0.042	0.212	0.233	0.968
$A_\beta/\sqrt{\Lambda}$	0.007	0.008	0.005	0.007
η_σ	0.561	0.710	0.789	0.936
η_ψ	0.066	0.040	0.027	0.007
$\bar{\eta}_\sigma$	0.541	0.729	0.765	0.971
$G_{\Lambda cr}$	9.989	5.325	3.613	1.006

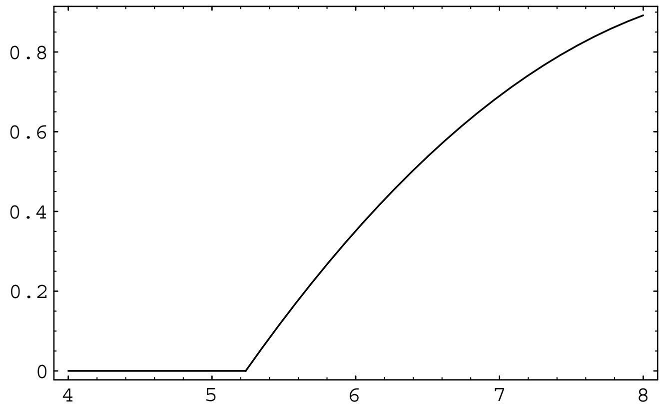

FIG. 1. Fermion-antifermion condensate as a function of the $(\bar{\psi}\psi)^2$ coupling G_Λ, in the range $[3/4G_{\Lambda cr}, 3/2G_{\Lambda cr}]$.

$$\hat{\nu} = 0.502, \quad \hat{\gamma} = 0.295, \quad \eta_\sigma = 0.710, \quad \eta_\psi = 0.040. \tag{18}$$

These values agree well with Eq. (17). In the opposite regime, $k \ll \bar{m}_R$, the running of the renormalized mass is only due to the anomalous dimension η_σ which is now different from the value (18). We find $\hat{\nu} = 0.500, \hat{\gamma} = 0.000, \eta_\sigma = 1.000, \eta_\psi = 0.000$. Again, these values agree well with Eq. (17) and the expectation $\hat{\gamma} = 0$. The nontrivial exponents $\hat{\nu}$, η_σ in the NY language correspond to the absence of renormalization effects for $\bar{G}_k$ for $k \to 0$ in the GN language. We note that for fixed δe the renormalized scalar mass (which corresponds to the inverse correlation length) scales as $m_R \sim k$ for $m_R \ll k$ and $m_R \sim \sqrt{k}$ for $m_R \gg k$. The value of $\eta_\sigma = 1$ for $k \ll m_R$ corresponds to η_2.

We also have computed the (nonuniversal) critical amplitudes which describe the dependence of $\bar{m}_R$ and $\bar{m}$ on the coupling G_Λ of the GN model. Observing $\delta e/e_{\Lambda cr} = G_{\Lambda cr}\delta(1/G_\Lambda)$ we obtain, for small deviations from criticality,

$$\bar{m}_R = A_\nu \left| \frac{\delta G_\Lambda}{G_{\Lambda cr}} \right|^\nu, \qquad \bar{m}^2 = A_\gamma \left| \frac{\delta G_\Lambda}{G_{\Lambda cr}} \right|^\gamma. \tag{19}$$

In the low temperature phase the running of σ_0 stops for small k and the complications of the symmetric phase are absent. With $\sigma_0 = \lim_{k\to 0} \sigma_{0k}$ one finds for small δe

$$\sigma_0 = A_\beta \left| \frac{\delta G_\Lambda}{G_{\Lambda cr}} \right|^\beta. \tag{20}$$

Our results, together with the scaling relation $\beta = \frac{\nu}{2}(d - 2 + \eta_\sigma)$, are reported in Table I.

In Fig. 1 we plot the condensate σ_0 as a function of G_Λ.

For all $N \geq 2$ the existence of a second order phase transition is confirmed by our analysis. As can be checked, the scaling relations are well verified. To compare with existing results obtained in the $1/N$ expansion, let us fix $N = 12$. In [6] the critical exponents have been calculated to the order $1/N$ yielding $\nu = 1.022$, $\gamma = 1.068$, $\beta = 1$, $\eta_\sigma = 0.955$. In the same paper Monte Carlo simulations for $N \geq 12$ are also reported. Conformal techniques have been used to calculate the anomalous dimensions to $O(1/N^3)$ [5] and yield $\eta_\psi = 0.013$, $\eta_\sigma = 0.913$. In [10,11] the universality class of the GN model is investigated numerically. For $N = 2$ [10] they find $\nu = 1.000$, $\gamma/\nu = 1.246$, $\beta/\nu = 0.877$, and obtain compatible results for the NY model through a second order epsilon expansion. For $N = 4$ [11] they find $\nu = 1.02$, $\gamma/\nu = 1.19$, $\beta/\nu = 0.89$, and obtain consistent results for the NY model at sufficiently strong Yukawa coupling.

The case $N = 1$ appears to be different from $N > 1$. We find a phase transition. For small G_Λ (large e_Λ) Eq. (9) is valid ($N' = 2$) and G_k scales according to its canonical dimension. The model is in the symmetric phase. For $G_\Lambda > G_{\Lambda cr}$, $G_{\Lambda cr} = 19.416$ the mass term at the origin of the potential becomes negative, indicating spontaneous symmetry breaking. We find no scaling solution, neither for $e_k \geq 0$ nor for $\kappa_k \geq 0$. This may suggest a first order transition.

In conclusion, a simple truncation of the exact flow equation for the effective average action gives a consistent picture for a second order phase transition for the GN model with $N \geq 2$ in three dimensions. We have computed critical exponents and amplitudes, and we relate directly physical observables like the correlation length or the order parameter to the value of the coupling G. By choosing different initial conditions we have also explicitly verified that the Neveu-Yukawa model belongs to the same universality class as the Gross-Neveu model.

*Alexander von Humboldt fellow

[1] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
[2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1989).
[3] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).
[4] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. D **39**, 3088 (1989).
[5] J. A. Gracey, Int. J. Mod. Phys **A9**, 727 (1994); A. N. Vasilev, S. E. Derkachev, N. A. Kivel, and A. S. Stepanenko, Theor. Math. Phys. **94**, 127 (1993).
[6] S. Hands, A. Kocić, and J. B. Kogut, Ann. Phys. (N.Y.) **224**, 29 (1993).
[7] C. Wetterich, Phys. Lett. B **301**, 90 (1993); J. Berges, N. Tetradis, and C. Wetterich, hep-ph/0005122.
[8] C. Wetterich, Z. Phys. C **58**, 585 (1993).
[9] N. Tetradis and C. Wetterich, Nucl. Phys. **B422**, 541 (1994).
[10] L. Karkkäinen, R. Lacaze, P. Lacock, and B. Petersson, Nucl. Phys. **B415**, 781 (1994).
[11] E. Focht, J. Jersák, and J. Paul, Phys. Rev. D **53**, 4616 (1996).