Limitations in Using Luminosity Distance to Determine the Equation of State of the Universe

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Supernova searches have been been suggested as a method for determining precisely the current value and time variation of the equation of state, w, of the dark energy component responsible for the accelerated expansion of the Universe. We show that the method is fundamentally limited by the fact that luminosity distance depends on w through a multiple integral relation that smears out information about w and its time variation. The effect degrades the resolution of w that can be obtained from current data.

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Recent observations suggest that most of the energy density of the Universe consists of a dark energy component with negative pressure that causes the expansion rate of the Universe to accelerate [1]. A key challenge for cosmology and for fundamental physics is to determine the nature of the dark energy. One possibility is that the dark energy consists of vacuum energy or cosmological constant, in which case the equation of state is $w \equiv p/\rho = -1$, where p is the pressure and ρ is the energy density of the dark energy. An alternative is quintessence [2], a time-evolving, spatially inhomogeneous energy component with negative pressure, such as a scalar field slowly rolling down a potential. For quintessence, the equation of state is typically a function of redshift, $w(z)$, whose value differs from -1. Hence, a precise measurement of w today and its time variation could distinguish between the two possibilities and provide important clues about the dynamical properties of dark energy.

Searches for type Ia supernovae at deep redshift have provided the most direct evidence that the expansion rate of the Universe is accelerating [3,4]. The supernovae appear to be standard candles which can be used to measure the luminosity distance-redshift relation. By measuring 50 supernovae out to redshift near $z = 1$, the Supernovae Cosmology Project (SCP) [3] and the High-z Survey [4] Project have each found strong evidence that the Universe is accelerating and that the equation of state of the dark energy component is negative [5].

A supernova search extended to greater z can make a much more precise determination of the luminosity distance as a function of redshift [6,7], $d_L(z)$, perhaps to better than 1% uncertainty out to redshift $z = 2$. (1% is probably an optimistic estimate of the limiting systematic uncertainty.) Does this enable a precise determination of the equation of state of the dark energy component and its time variation? As we show in this paper, the answer is no. The inherent limitation is theoretical: the luminosity distance depends on $w(z)$ through a multiple-integral relation that smears out detailed information about $w(z)$. Consequently, the value of $w(z)$ today is poorly resolved and no useful constraint can be obtained about its time variation.

The problem can be immediately appreciated from Fig. 1, which compares $d_L(z)$ for an assumed cosmological model [$\Omega_m = 0.3$, $\Omega_Q = 0.7$, and $w_Q = -0.7 =$ const, where $\Omega_{m,Q}$ is the ratio of the (matter, quintessence) energy density to the critical density] with eight other models chosen as examples where $d_L(z)$ is nearly degenerate with the assumed model. In this figure and throughout the paper, we assume the Universe is cosmologically flat and the speed of light $c = 1$. (Henceforth, we use the subscript Q to label the dark energy component, be it quintessence or cosmological constant.) Figure 1a shows that $d_L(z)$ is nearly identical for the set of models as individual curves can hardly be distinguished. Figure 1b displays the percentage deviation of $d_L(z)$ from the assumed model, where it can be seen that the deviation is less than 1% out to redshift $z = 2$. Figure 1c then shows $w_Q(z)$ for the respective models. The striking result is the wide range of $w_Q(z)$ that produces nearly the same $d_L(z)$ as the assumed model. If one expands $w_Q(z) = w_0 + w_1 z + w_2 z^2 + \ldots$, then, for this particular collection of models, w_0 varies between -0.55 and -0.9 (a total span of 50% about the assumed value, $w_Q = -0.7$) and $w_1 = dw_Q/dz_0$ varies between -1.1 and $+1.6$. (The subscript "0" refers to present-day values of parameters.)

Note that the degenerate models chosen for the illustration span a larger range of $|dw_Q/dz| = \mathcal{O}(1)$ than most realistic models predict. Typically, $|dw_Q/dz_0| \ll 1$ because $w_Q(z)$ is bounded in most cases to lie between -1 and $+1$ in order that the dark energy obey the positive energy condition and be stable under perturbations. The large uncertainty in $w_1 = dw_Q/dz_0$ means that little useful information is obtained about the magnitude or sign of the time variation of w_Q. Also, w_0 is poorly resolved. The resolution of $w_Q(z)$ degrades significantly further if one includes the uncertainty in Ω_m, as shown in Fig. 2 (see discussion below).

Our conclusion may seem at odds with some projections of what can be obtained in future supernova searches [6,7]. Many analyses assume $w_Q =$ const. If we impose this condition, then the range of models that fit collapses to the narrow region between the dashed lines in Fig. 1c, giving

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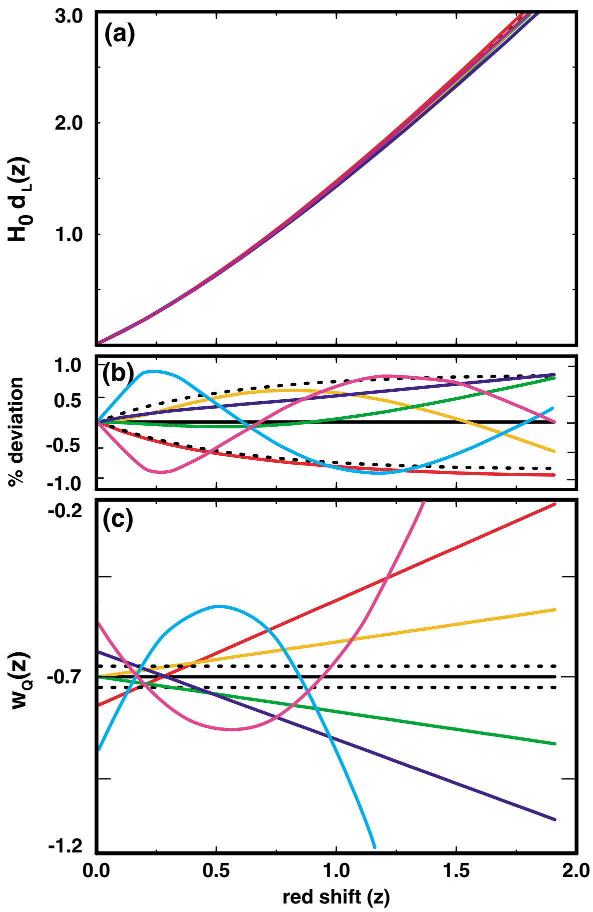

FIG. 1 (color). (a) The luminosity distance $H_0 d_L(z)$ for nine choices of equation of state $w_Q(z)$ for the dark energy shown in (c), where H_0 is the current value of the Hubble parameter. All models have $\Omega_m = 0.3$. (b) Illustrates that the percentage deviation of $d_L(z)$ from a cosmological model with $\Omega_m = 0.3$, $\Omega_Q = 0.7$, and $w_Q = -0.7 = \text{const}$ is less than 1%. If one artificially restricts w_Q to be constant, then the range of models collapses to the region between the dashed lines.

a misleading impression that $w_Q(z)$ is well resolved. However, if consideration is extended to models in which w_Q is z dependent, such as the linear form $w_Q(z) = w_0 + w_1 z$, the result is dramatically different. A very wide range of (w_0, w_1) produces nearly identical $d_L(z)$ because the differences are smoothed out by the multiintegral relation derived below between luminosity distance and $w_Q(z)$. This degeneracy accounts for the results found in Figs. 1c and 2, but is missed if one artificially restricts w_Q to be constant. Notice in Fig. 1c that including nonlinear forms for $w_Q(z)$ enhances the uncertainty in w_0 and w_1 even further. Among studies which have considered time-varying $w_Q(z)$, our results agree with some [8–10] but seem significantly less optimistic than others [7,11]. In the latter cases, the assumptions about the observations are similar but various subtle factors, such as the use of fitting functions rather than exact expressions for $d_L(z)$ or imposing the constraint $w_Q > -1$, combine numerically to reduce artificially the degeneracy. Extending searches to yet deeper redshift ($z > 2$) does not help either because the effect of quintessence on $d_L(z)$ is proportional to Ω_Q which becomes very small at deep redshift.

The key to understanding these conclusions is the relation between luminosity distance and the equation of state. The luminosity distance is related to the Robertson-Walker scale factor $a(t)$ through the equation

$$d_L(z) = (1 + z)\int_a^{a_0} \frac{da'}{\dot{a}'a'} = (1 + z)\int_1^{1+z} \frac{dx}{H}, \quad (1)$$

where the redshift z satisfies $1 + z \equiv a_0/a$, and H is the Hubble parameter, $H^2 = H_0^2[\rho_T(z)/\rho_T(0)]$. We assume a flat universe. The subscript "T" refers to the total equation of state of the combined matter-quintessence fluid. Integrating the energy conservation equation,

$$\dot{\rho}_T = -3H(1 + w_T)\rho_T \quad (2)$$

we can reexpress $H^2 = H_0^2[\rho_T(z)/\rho_T(0)]$ as

$$H^2 = H_0^2 \frac{\rho_T(z)}{\rho_T(0)} = H_0^2 \exp\left[3\int_1^{1+z} d\ln x(1 + w_T)\right]. \quad (3)$$

and the luminosity distance as

$$d_L(z) = \frac{1 + z}{H_0}\int_1^{1+z} dx' \times \exp\left[-\frac{3}{2}\int_1^{x'} d\ln x(1 + w_T)\right]. \quad (4)$$

Equation (4) shows that the luminosity distance depends on a double integral over the *total* equation of state, $w_T(z)$. One integral is required to obtain the total luminosity distance from the present back to redshift z. The integrand depends on H which is itself related to w_T through the integral relation, Eq. (3). To distinguish different forms of dark energy, though, we need to determine $w_Q(z)$, the equation of state of the dark energy component. Assuming that the Universe contains only pressureless matter (baryonic and cold) and dark energy, then $w_T = w_Q\Omega_Q$, where Ω_Q is itself related to $w_Q(z)$ through an integral relation. In particular, using the energy conservation analogous to Eq. (2) for the dark energy component alone (p_Q, ρ_Q), one obtains

$$\frac{\rho_Q(z)}{\rho_Q(0)} = \exp\left[3\int_1^{1+z} d\ln x(1 + w_Q)\right]; \quad (5)$$

combined with Eq. (3), $w_T = w_Q\Omega_Q = w_Q\rho_Q/\rho_T$ can be reexpressed as

$$w_T(z) = w_Q(z)\left\{1 + \frac{\Omega_m}{\Omega_Q}\exp\left[-3\int_1^{1+z} d\ln x(w_Q)\right]\right\}^{-1}, \quad (6)$$

where $\Omega_{m,Q}$ refers to the current values. Together with Eq. (4), this expression constitutes the integral relation between luminosity distance and $w_Q(z)$ that underlies the degeneracy problem.

To express the degeneracy problem quantitatively, we have found the maximum likelihood values of w_0 and w_1

based on current SCP supernova data [3], which has measured 50 supernova out to redshift $z \approx 1$. For simplicity, we have assumed $w_Q(z) = w_0 + w_1 z$; including more general functions of z only degrades the resolution further. Furthermore, we have repeated the analysis based on simulated data from an idealized experiment which measures thousands of supernovae out to redshift $z = 2$. The simulated data assume a cosmological model with $\Omega_m = 0.3$, $\Omega_Q = 0.7$, and $w_Q = -0.7 = \text{const}$. For the idealized experiment, the absolute magnitude measurement for a single supernova is taken to have a statistical variance of 0.15 and a systematic measurement error of 0.02. The supernovae are divided into 50 bins between $z = 0$ and $z = 2$ such that the statistical and systematic errors averaged over a bin are comparable, resulting in an error in $d_L(z)$ for a given bin equal to 0.6%. We assume that other types of observations have constrained Ω_m to lie between 0.2 and 0.4, say, and marginalize Ω_m over that range. For the purposes of illustration, we assume that $w_Q(z)$ is a linear function of z parametrized by w_0 and w_1. As shown in Fig. 1c, including more general forms for $d_L(z)$ only worsens the degeneracy problem.

We have considered two ways of treating the systematic errors. For case I, we assume that systematic errors are random and uncorrelated from bin to bin and perform a likelihood analysis over the 50 bins with 0.6% error each to determine the uncertainty in $d_L(z)$. (To obtain 0.6% per bin for 50 bins requires measuring thousands of supernovae.) In case II, we assume there is negligible statistical uncertainty but correlated systematic error of 1%. Examples of case II errors are those due to calibration, dust, or evolution of supernovae. In this case, all models which predict $d_L(z)$ within 1% of the assumed cosmological model for all z between 0 and 2 are deemed indistinguishable. As it turns out, the two-sigma likelihood contours in case I are roughly equivalent to the indistinguishability region of case II, so both cases give comparable results.

For the current data, likelihood analyses based on the assumption that w_Q is constant have reported a resolution of $-1 < w_Q < -0.6$ at the 95% level [5]. When we repeat the analysis assuming a linear form for $w_Q(z)$ and making no prior assumptions about w_0 and w_1, we find that neither parameter is well determined. The degeneracy obliterates the resolution of both quantities: w_0 can vary between -3.2 and -0.4 (95% confidence) and w_1 can vary between -11.8 and 11.0. Notice the enormous range of w_0; the 99% confidence contour includes positive values, so one cannot even be sure that the Universe is accelerating today. In cases where the Universe is not accelerating today, we can still conclude that it must have been accelerating recently because w_1 is highly negative whenever w_0 is positive. One could argue that allowing large values of w_1 so that $w_Q(z)$ becomes much less than -1 or greater than $+1$ between $z = 0$ and $z = 1$ (the range of current observations) is unphysical based on the positivity and stability conditions that apply to most (but not all) forms of dark energy. Adding this theoretical constraint, the two-sigma range for w_0 is found to lie between -0.5 and -1.0 (95% confidence), in which case one may conclude that the Universe is accelerating today. However, one should beware that our estimates are optimistic in assuming that $w_Q(z)$ has only a linear and constant term. A safer assessment would be that, assuming positivity and stability but no other prior about w_Q, one can conclude from present data that the Universe is accelerating today, but w_Q is very poorly resolved, and dw_Q/dz can range anywhere within the imposed positivity and stability constraints.

For the idealized experiment, the likelihood contours span a substantial range of (w_0, w_1), as shown in Fig. 2. In the shaded ellipses, the figure shows likelihood region if one assumes prior knowledge that $\Omega_m = 0.3$ precisely. The contours stretch along a curve in the $w_0 - w_1$ plane which corresponds to a near degeneracy. It is this degeneracy that dashes hopes of using luminosity distance to measure both the current value and time derivative of w_Q. Marginalizing over Ω_m expands the contours along a direction nearly orthogonal to the degeneracy curve. See the large black contours in Fig. 2. Within the 95% confidence region, w_Q spans a range equal to more than 35% of the assumed value ($w_Q = -0.7$), and $w_1 = dw/dz_0$ ranges between $+0.3$ and -1.1. Expanding the fit to include nonlinear $w_Q(z)$ would expand the region even more (see Fig. 1).

Another approach for measuring the time evolution of $w_Q(z)$ is object counts, where the objects might be galaxies, clusters, or halos: $\frac{dN}{d\Omega} = n_c r^2 dr$, where dN

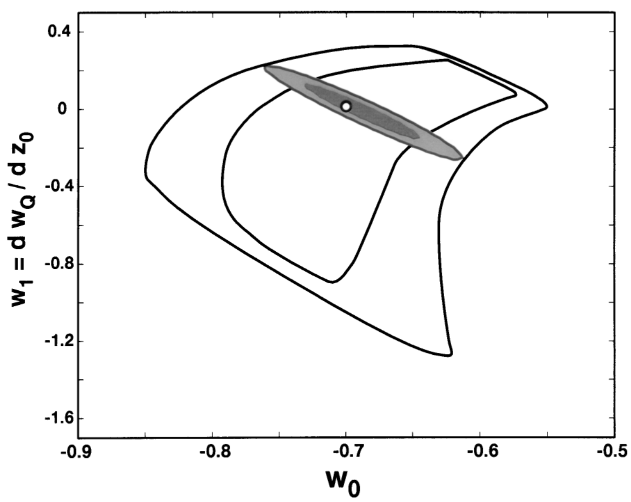

FIG. 2. One- and two-sigma contours in the $(w_0, w_1) \equiv (w_Q(z = 0), dw_Q/dz_0)$ plane for an idealized experiment which measures thousands of supernovae between $z = 0$ and $z = 2$. The supernovae are divided into 50 bins with a net error of 0.6% per bin. The example assumes a model with $\Omega_m = 0.3$, $\Omega_Q = 0.7$, $w_Q = -0.7 = \text{const}$, indicated by the circle. The thin shaded ellipses are one- and two-sigma contours if one assumes Ω_m is fixed to be precisely 0.3. The broader black contours are the result if Ω_m is marginalized over the range 0.2 to 0.4.

is the number of objects in a comoving volume element $r^2 dr d\Omega$ for coordinate distance r and solid angle Ω. One assumes that the number density of objects per comoving volume, n_c, is constant or some known function of z. The distance r is related to the luminosity distance by $d_L(z) = (1 + z)r$ (if we normalize the Friedmann-Robertson-Walker scale to be unity today). Hence, we can write

$$\frac{dN}{dz d\Omega} = n_c r^2 \frac{dr}{dz} = n_c \frac{d_L^3}{(1 + z)^4} \left[\frac{(1 + z) d_L'}{d_L} - 1 \right], \tag{7}$$

where prime represents derivative with respect to z. The novel feature here is $d_L'(z)$, which entails one less integral of w_Q than d_L, and, hence, perhaps an improved resolution. Newman and Davis [12] suggest that near-future observations of the number of dark matter halos as a function of their circular velocity and redshift can determine $\frac{dN}{dz d\Omega}$ to within a few percent. Assuming a resolution of 2.5% between $z = 0.7$ and $z = 2$ (more optimistic than their example), we find combinations of w_0 and w_1 for which $dN/dz d\Omega$ is indistinguishable from the assumed cosmological model. The indistinguishability region coincides approximately with the two-sigma contours in Fig. 2. Hence, object counts are subject to essentially the same degeneracy problem as supernova searches.

Our analysis has shown that the luminosity distance-redshift relation and similar classical cosmological measures are limited in their ability to resolve $w_Q(z)$, even under optimistic assumptions (an absolutely flat universe, rather stringent priors for Ω_m, exceptional accuracy in determining luminosity distance, etc.) These conclusions hold unless the errors can be reduced by at least 1 order of magnitude or some complementary experiment can break the degeneracy. If a method could be found to reduce considerably the uncertainty in Ω_m—in Fig. 2, we assumed Ω_m to be in the range 0.2 to 0.4—the degeneracy region shrinks along one direction. This would improve the resolution, but only modestly because there remains the second degeneracy direction shown in the figure. It should be noted that reducing the uncertainty in Ω_m will be difficult. Most methods for determining Ω_m are dependent on some assumption about w_Q. In the case of the CMB anisotropy, for example, a degeneracy arises such that, for the same high-precision data, one can derive different values of Ω_m depending on what assumption is made about $w_Q(z)$ [13]. Of course, if one assumes $|dw_Q/dz| \ll 1$ (or some other prior), then deep supernova searches and galaxy counts can resolve $w_Q(z = 0)$ with impressive precision, but the value depends strongly on the particular theoretical assumption.

Our conclusions also undermine claims that the supernova and object count searches can determine the future fate of the Universe. Since the observations cannot distinguish whether dw_Q/dz_0 is positive or negative, they cannot distinguish whether w_Q will remain negative or become positive in the future, and, hence, whether the Universe will accelerate ever faster or cease accelerating altogether.

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