Scars of the Wigner Function

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We propose a picture of Wigner function scars as a sequence of concentric rings along a two-dimensional surface inside a periodic orbit. This is verified for a two-dimensional plane that contains a classical hyperbolic orbit of a Hamiltonian system with 2 degrees of freedom. The stationary wave functions are the familiar mixture of scarred and random waves, but the spectral average of the Wigner functions in part of the plane is nearly that of a harmonic oscillator and individual states are also remarkably regular. These results are interpreted in terms of the semiclassical picture of chords and centers.

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Sixteen years have passed since Heller [1] detected scars of periodic orbits in individual eigenfunctions of chaotic systems. Explanations in terms of wave packets [1] or the semiclassical Green function [2,3] do predict an enhancement of intensity near the projection of a Bohr-quantized periodic orbit. However, such theories apply to a collective superposition of states near this quantization condition. In spite of some quite striking visual evidence, the issue of scarring for individual states is confused by the fact that different branches of the same periodic orbit may overlap so that their contributions interfere in the position space and this must be superposed on the expected random wave background for the chaotic state. This has led to the need to make quantitative assessments of scarring strength [4]. We here show that the phase space picture can provide sharper qualitative evidence of the influence of a periodic orbit on its Wigner functions.

The old conjecture of Berry and Voros [5] that the Wigner functions for eigenstates of chaotic Hamiltonians are concentrated on the corresponding classical energy shells is widely disseminated [6], as it is compatible with Schnirelman's theorem [7]. Nonetheless, no such restriction arises in the more recent semiclassical theory for the mixture of these states over a narrow energy window, i.e., the spectral Wigner function [8,9]. Indeed, the computations presented in this Letter show that individual Wigner functions, as well as their energy average can oscillate with large amplitudes deep inside the energy shell.

At points $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ inside the energy shell, the semiclassical spectral Wigner function is [8,9]

$$W(\mathbf{x}, E, \varepsilon) = \sum_j A_j e^{-\varepsilon t_j/\hbar} \cos\left[\frac{S_j(\mathbf{x}, E)}{\hbar} + \gamma_j\right]. \quad (1)$$

The sum is over all the trajectory segments on the E shell with end points, $\mathbf{x}_{j\pm}$, centered on $\mathbf{x}$, as sketched in Fig. 1. The action is merely the symplectic area, $S_j = \oint \mathbf{p} \cdot d\mathbf{q}$, for the circuit taken along the orbit from $\mathbf{x}_{j-}$ to $\mathbf{x}_{j+}$ and closed by the chord $-\boldsymbol{\xi}_j(\mathbf{x})$. The time of traversal for the stretch along the trajectory is t_j and ε is the width of the energy window over which we average individual Wigner functions $W_n(\mathbf{x})$, i.e.,

$$W(\mathbf{x}, E, \varepsilon) = (2\pi\hbar)^{D/2} \sum_n \frac{\varepsilon/\pi}{(E - E_n)^2 + \varepsilon^2} W_n(\mathbf{x}), \quad (2)$$

where D is the dimension of the phase space $(\mathbf{q}, \mathbf{p})$. We shall not be concerned with the Maslov phase γ_j or with the amplitude $A_j(\mathbf{x}, E)$, except to note that these are purely classical quantities that vary smoothly with $\mathbf{x}$ inside the shell, as compared with the high frequency oscillations of the cosine factor.

The energy shell itself is a caustic, because the chord $\boldsymbol{\xi} \to 0$ as $\mathbf{x}$ approaches the shell. In this limit, the contributing trajectories either shrink to a point or they are very close to a periodic orbit. The modification of (1) leads to the Berry theory of scars [10], further refined in [8], for evaluation points $\mathbf{x}$ close to the energy shell.

The point here is that large open segments of periodic orbits can be constructed by adding multiple windings to a primitive, small segment of a periodic orbit. If conditions for phase coherence, to be discussed, are satisfied, we can then obtain a scar deep inside the energy shell. This scar is located at the two-dimensional *central surface* constructed by the centers of all the chords with end points on the periodic orbit. For the case where the phase space

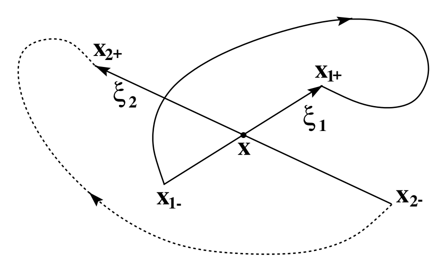

FIG. 1. In general, there are many orbit segments with end points $\mathbf{x}_{j-}$ and $\mathbf{x}_{j+}$ centered on a given point $\mathbf{x}$. The circuit, closed by the chord $-\boldsymbol{\xi}_j = \mathbf{x}_{j-} - \mathbf{x}_{j+}$, defines the phase of the semiclassical contribution to (1).

dimension $D = 2$, the central surface is the phase space region enclosed by the convex hull of the orbit that here is the energy shell. For $D \geq 4$, the two-dimensional central surfaces are still bounded by one-dimensional periodic orbits that lie within the higher dimensional energy shell. In the simplest case where the orbit is plane and convex, the central surface coincides with the part of the plane within the orbit just as in the case where $D = 2$. If the orbit is not plane, then the central surface will be more complicated and it may exhibit singularities and self-intersections. In general, it will not coincide everywhere with the invariant surface formed by changing continuously the energy of this periodic orbit.

We emphasize that these scars of the Wigner function that correspond to a very recognizable oscillatory pattern have special localization properties in phase space of dimension $D \geq 4$. Previous explorations of phase space representations of eigenstates have been too blunt to discern these fine features. The work of Feingold *et al.* [11] involves a projection from a three-dimensional section, which is fine for the Husimi function, but washes away the details of an oscillatory Wigner function. On the other hand, Agam and Fishman [12] restrict their investigation to the neighborhood of the orbit and, hence, to the energy shell. This is just the edge of the central surface.

In this Letter we tested our prediction for the simplest case of an unstable periodic orbit lying in a two-dimensional plane. Accidentally this plane is a symmetry plane; however, we have checked that the enhanced amplitude of the spectral Wigner function accompanies the distortion of the orbit due to a nonlinear canonical transformation. Since the Wigner function is not invariant in this case, we thus obtain an essentially new system, in which the new central surface is not a symmetry surface. Thus, our scarring effect cannot be attributed to a special property of the symmetry plane. This case is essentially different from recent studies of higher dimensional systems where the scars arise over marginally stable invariant planes that are indeed special [13].

The chord structure for segments of a periodic orbit is sketched in Fig. 2. This is similar to a system with $D = 2$, for which Berry [14] showed that there is only one chord for most points $\mathbf{x}$. However, we must now distinguish between the chords $\boldsymbol{\xi}_{\text{in}} = -\boldsymbol{\xi}_{\text{out}}$, and the sum in (1) includes each different winding of the periodic orbit for primitive orbit segments, "in" and "out," associated with the two respective chords. Evidently all these "in" and "out" contributions build up if the energy is close to one of the values for which this periodic orbit is Bohr quantized. We show in [15] that the condition for these two contributions to be in phase is the same as that for a maximal contribution to the Gutzwiller trace formula [6]. Therefore, near the quantization energy, the sum of all the contributions in (1), owing to chords in the periodic orbit, can be approximated by an expression analogous to Berry's simplest semiclassical approximation [14,16] for the Wigner function in $D = 2$

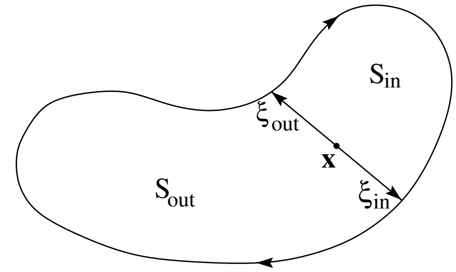

FIG. 2. If the end points lie on a periodic orbit, we can add an infinite number of windings to the two shortest primitive segments, "in" and "out," divided by the chords $\boldsymbol{\xi}_{\text{in}} = -\boldsymbol{\xi}_{\text{out}}$. All the points $\mathbf{x}$, centers of chords in the periodic orbit, form a two-dimensional *central surface*. If the orbit is plane (as in the graph) the central surface is the part of the plane enclosed by the convex hull of the orbit.

systems, but now evaluated over points $\mathbf{x}$ in the two-dimensional central surface. Hence, in this case, we have an enhanced contribution to the spectral Wigner function with the phase $S_{\text{in}}(\mathbf{x})/\hbar$ and the Maslov phase $\gamma = 0$. The successive contours $S_{\text{in}}(\mathbf{x}) = \text{const}$ thus determine rings of constant phase along the central surface. Therefore, we predict rings of positive and negative amplitude superimposed on the background of contributions from uncorrelated trajectory segments that also contribute to the spectral Wigner function. The edge of this system of rings along the orbit itself is the object of the Berry theory [10], though it does not deal with multiple windings.

We have tested this prediction for the Nelson Hamiltonian ($D = 4$) [17,18],

$$\mathrm{H}(\mathbf{q},\mathbf{p}) = (p_1^2 + p_2^2)/2 + 0.05q_1^2 + (q_2 - q_1^2/2)^2. \tag{3}$$

Evidently the plane $q_1 = p_1 = 0$ is classically invariant, so we obtain an isolated periodic orbit on this plane for each energy (the vertical family). This is then, the simplest case where the central surface is flat—the central plane. The restriction of the Hamiltonian to this plane defines a harmonic oscillator, so there is a single chord with end points in the periodic orbit (except for sign) for each point (p_2, q_2) inside the orbit, except at $q_2 = p_2 = 0$.

Using the method of [18] we computed the wave functions $\langle \mathbf{q} \mid n \rangle$ for the eigenstates of (3) within the range of energies $0.821 \leq E \leq 0.836$ taking $\hbar = 0.05$. The Wigner function for each state was then calculated directly by the double integral over $\mathbf{Q} = (Q_1, Q_2)$:

$$W_n(\mathbf{q},\mathbf{p}) = \int \frac{d\mathbf{Q}}{(2\pi\hbar)^2} \left\langle \mathbf{q} + \frac{\mathbf{Q}}{2} \,\middle|\, n \right\rangle \left\langle n \,\middle|\, \mathbf{q} - \frac{\mathbf{Q}}{2} \right\rangle \times e^{-i(\mathbf{p}\cdot\mathbf{Q}/\hbar)}. \tag{4}$$

In Fig. 3 we display the average over Wigner functions for the energy window chosen, surrounding the 11th Bohr energy for the vertical orbit that has Maslov index 3. This is more convenient than smoothing with the Lorentzian window in (2) and should produce essentially the same results.

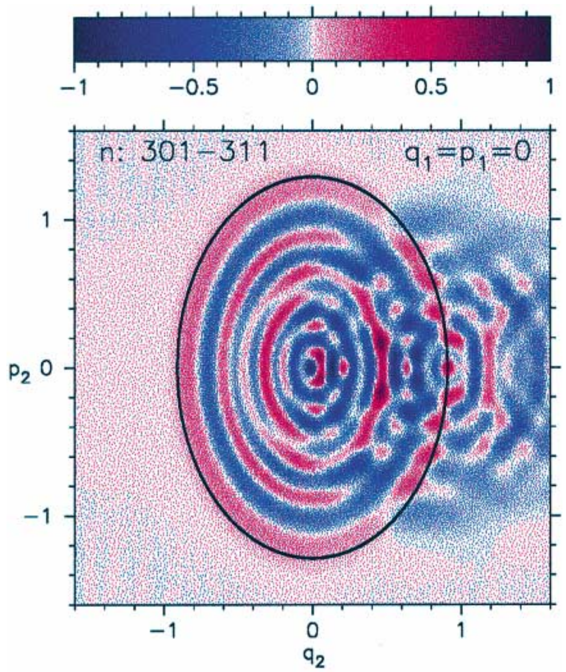

FIG. 3 (color). Averaged Wigner function on the (q_2, p_2) plane for an energy window containing 11 eigenstates centered on the 11th Bohr level. The thick dark line is the superposition of each energy shell in the plane (periodic orbit) for energies in the range considered; the middle one corresponding to the Bohr-quantization energy.

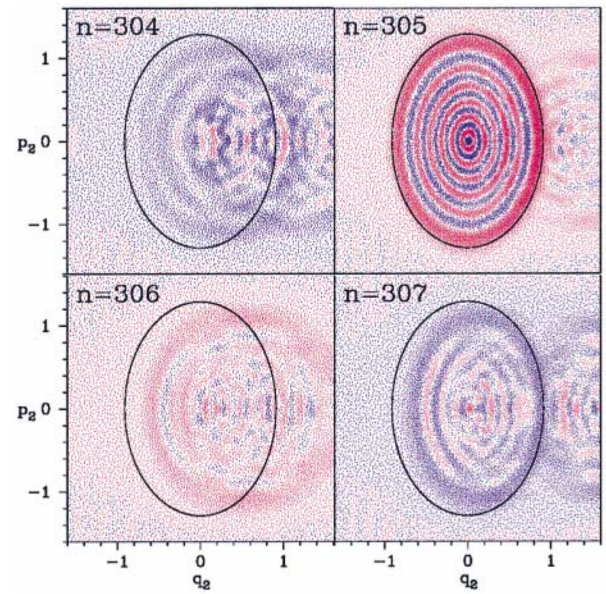

FIG. 4 (color). Some of the individual Wigner functions on the (q_2, p_2) plane that were averaged in Fig. 3. The plots are globally normalized to point out the enhanced amplitude of the scar for the $n = 305$ state. The eigenenergy of this state is at a distance $\approx 1.5\Delta$ from the Bohr energy (Δ the mean level spacing in the chosen average window).

As an example, Fig. 4 shows some of the individual Wigner functions in this window for $\mathbf{x}$ in the (q_2, p_2) plane. Immediately, we notice that all these Wigner functions are remarkably regular, roughly in the region $q_2 < 0$. Indeed, Fig. 3 coincides in this region with the Wigner function of the restricted Hamiltonian, also with respect to the precise phase of the oscillations. The individual Wigner functions follow the same phase contours, but there is a varying phase shift. If we extrapolate the semiclassical theory to energy smoothings of the order of the energy spacing, the conclusion is that there are no orbit segments, other than the periodic orbit itself, contributing to the Wigner functions on this surface up to the Heisenberg time. This is more dramatic for the state $n = 305$ whose Wigner function on all the central surface is almost the same as that of the restricted Hamiltonian (see Fig. 4). It is important to note that only this state seems to have a visual scar of the vertical orbit in position space, although superimposed to a random wave background.

For $q_2 > 0$, we have found other orbit segments that do not lie on the central plane, but which do contribute to the spectral Wigner function on it. Indeed, this is the case for segments of both the symmetric periodic orbits shown in Fig. 5 projected onto position space. Needless to say, the multiple windings of these periodic orbits could produce total contributions of the same order as the plane orbit, if they are nearly Bohr quantized. This supplies a qualitative explanation for the existence of interference patterns to the left of the Wigner functions in Figs. 3 and 4. Notice that the second of the orbits in Fig. 5 reaches into a region with q_2 greater than is attained by the plane periodic orbit. So we account for Wigner functions reaching outside the energy shell (which coincides with the periodic orbit on the central plane). Note that these are only particular examples of the many symmetric periodic orbits whose central surface intersects the central plane along lines with $q_2 > 0$ [17,18]. Therefore, it will be hard to make quantitative predictions in this region in contrast to that where $q_2 < 0$, for which no such orbits of short period were found. The enhanced scar pattern of the Wigner function of the state $n = 305$ over all the central plane evidently washes out other possible contributions.

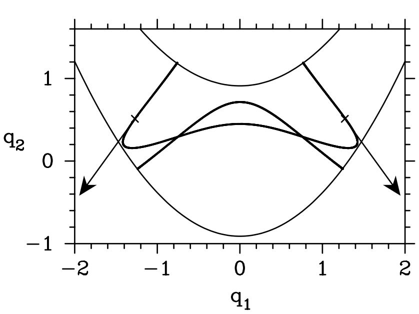

FIG. 5. Two of the many symmetric periodic orbits that have chords centered on the (q_2, p_2) plane. The graph is in position space, so that the momenta for the symmetric end points are displayed as vectors. All centers have $q_2 \gtrsim 0$. The thin line is an equipotential of (3) for the energy of the 11th Bohr level.

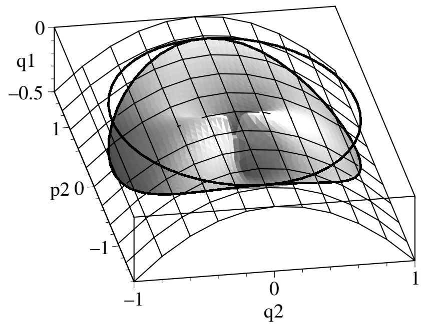

FIG. 6. The application of a particular nonlinear canonical transformation to the Hamiltonian (3) deforms the invariant plane $q_1 = p_1 = 0$ into the squared surface in the graph that also has $p_1 = 0$. The curve on this surface is the distortion of the original periodic orbit (ellipse displayed in the invariant plane). The grey surface is the new central surface for the deformed periodic orbit.

We show that the symmetry of the central plane is not responsible for the scarring observed by applying a nonlinear canonical transformation, described in [15], that deforms it into a new invariant surface (Fig. 6). Now the new central surface of the distorted periodic orbit no longer coincides everywhere with the invariant surface (Fig. 6), losing in this way any symmetry. We calculated the Wigner functions for this new system, over the central surface of the periodic orbit, for eigenstates within the same range of energies surrounding the 11th Bohr level. Both the average and the individual Wigner functions, projected over the (q_2, p_2) plane, display the same features as those shown in Figs. 3 and 4, respectively. Reference [15] also considers the thickness of scar surfaces.

Our main point is that individual trajectory segments do play a role in the spectral Wigner function. Indeed, phase coherent contributions from a periodic orbit can add up to scars in the form of concentric rings deep inside the energy shell. Even though the Berry-Voros hypothesis [5] ties in more intuitively with the concept of quantum ergodicity, it should be noted that the chord picture of Wigner functions does not contradict Shnirelman's theorem and subsequent exact results [7], because the oscillations make negligible contributions to integrals with smooth functions, i.e., projections that give true probability densities. However, the oscillations are unavoidable in the study of interference effects.

It would seem that the structure of interfering chords, from which the spectral Wigner function is built up, could generate only a messy picture where scars would be even harder to recognize than in wave functions for which quantitative methods are needed. We have now shown that the converse can be true in the simplest case, i.e., that the relatively low dimension of the central surface can allow it to miss, over an appreciable region, the contribution of chords from other orbits. In this region we obtain a regular ring structure, not only for the energy average, but even for the pure Wigner functions shown in Fig. 4. Though we cannot yet predict why the relative weight of this particular periodic orbit should vary so markedly among the states close to the Bohr-quantized energy, we are now able to recognize in Fig. 4 a remarkable example of an individual scar.

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