Towards the Large N Limit of Pure $\mathcal{N} = 1$ Super Yang-Mills Theory

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We find the gravity solution corresponding to a large number of Neveu-Schwarz or D5-branes wrapped on a two sphere so that we have pure $\mathcal{N} = 1$ super Yang-Mills in the IR. The supergravity solution is smooth, it shows confinement, and it breaks the $U(1)_R$ chiral symmetry in the appropriate way. When the gravity approximation is valid the masses of glueballs are comparable to the masses of Kaluza-Klein (KK) states on the 5-brane, but if we could quantize strings on this background it looks like we should be able to decouple the KK states.

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Introduction.—The Anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] gives the large N dual description for $\mathcal{N} = 4$ super Yang-Mills. It would be nice to find similar correspondences for "pure" (without matter) Yang-Mills theories with less supersymmetry. In this paper we consider a little string theory [4], or Neveu-Schwarz (NS) 5-brane theory, in type-IIB string theory. In the IR this theory reduces to six-dimensional super Yang-Mills with 16 supercharges. We wrap this brane on S^2 and we twist the normal bundle in such a way that we preserve only $1/4$ of the supersymmetries and we give a mass to the four scalar fields. This theory reduces then to pure $\mathcal{N} = 1$ super Yang-Mills in the IR. We find the supergravity solution using methods similar to the one in [5], reading the solution from [6] and then lifting it up to ten dimensions using [7,8]. When the supergravity approximation is valid the little string theory scale and the scale of the four-dimensional theory are comparable. Nevertheless, the solution has all the expected qualitative features. It has a $U(1)_R$ symmetry broken in the UV to Z_{2N} and the full solution breaks it further to Z_2 and we find N different solutions. The theory is confining and it is magnetically screening. It has domain walls between the different vacua. Strings can end on the domain walls. When we try to take the decoupling limit we find a rather precise Ramond-Ramond (RR) σ model that we should quantize in order to find the decoupled string theory describing $\mathcal{N} = 1$ super Yang-Mills.

NS 5-branes on S^2.—Since the appropriate UV description of the 5-brane theory is the little string theory, or NS 5-brane, we start with an NS 5-brane in type-IIB string theory. The geometry dual to the little string theory is $ds^2_{\rm str} = dx_6^2 + N(d\rho^2 + d\Omega_3^2)$, $e^{\phi} = e^{\phi_0 - \rho}$, where ϕ_0 is an arbitrary constant that can be changed by shifting ρ. N is the number of 5-branes. This represents a 5-brane whose world volume is R^6. Now we would like to consider a brane whose world volume is $ds_6^2 = dx_4^2 + Ne^{2g} d\Omega_2^2$, so that the brane is wrapped on a two sphere of radius $R^2 = Ne^{2g}$. The factor of N is introduced just for later convenience. In order to preserve supersymmetry we should twist the normal bundle. Since the nontrivial part of the spin connection is in a U(1) subgroup we should choose how to embed the U(1) in $SO(4) \sim SU(2)_R \times SU(2)_L$. If we embed the spin connection in $U(1)_R \subset SU(2)_R \subset SO(4)$ we preserve only four supercharges or $\mathcal{N} = 1$ supersymmetry in four dimensions. Let us see this more precisely from the 5-brane world volume point of view. The spinors that generate the supersymmetries on the NS 5-brane are two six-dimensional spinors with positive chirality that are in the $(\mathbf{2}, \mathbf{1})$ of $SU(2)_R \times SU(2)_L$ and two negative chirality spinors in the $(\mathbf{1}, \mathbf{2})$. The supersymmetries that are generated by the spinors transforming under $SU(2)_L$ are broken. The preserved supersymmetries have positive chirality in six dimensions and are such that the $U(1)_R$ charge is correlated with the chirality of the spinor in the two directions of the sphere. We see that this leaves us with $1/4$ of the original supersymmetries of the 5-brane. The four scalars transverse to the 5-brane transform under the $(\mathbf{2}, \mathbf{2})$ of $SU(2)_R \times SU(2)_L$. This implies that, after twisting, they become spinors on the two sphere so that they do not have any zero modes. In the IR the only massless fields are the gauge fields and the gauginos. So in the IR we have pure $\mathcal{N} = 1$ super Yang-Mills. The value of the Yang-Mills coupling is given in terms of the volume of the sphere by $\frac{1}{g_4^2} = \frac{\mathrm{Vol}_{S^2}}{g_6^2} = \frac{Ne^{2g}}{2\pi^2}$. We are implicitly assuming that the volume of the S^2 is much larger than the five-dimensional gauge coupling.

This twisted 5-brane theory seems to have a $U(1)_R$ symmetry which is the $U(1)_R$ that we are twisting. This is the $U(1)_R$ symmetry of $\mathcal{N} = 1$ super Yang-Mills; it acts on the gluinos but not on the gauge fields. We will see that this $U(1)_R$ symmetry is broken to Z_{2N} by world sheet instantons in the NS description. This twisting also preserves the $SU(2)_L$ symmetry of the 5-brane theory. But this symmetry does not act on the massless fields, it acts only on the Kaluza-Klein modes which are expected to decouple in the IR.

Finding the gravity solution.—As explained in [5] we need to impose an appropriate boundary condition for the geometry. In this case the boundary is at $\rho \to \infty$. So we impose the condition that the seven-dimensional geometry has a boundary which is $R^4 \times S^2$ and we twist by imposing appropriate boundary conditions for the seven-dimensional gauge fields which come from the isometries of S^3. In this case we will set $A^3 = \cos\theta d\varphi$ for large ρ. It turns out that an ansatz like this is possible only if we allow the volume of S^2 to grow as $\rho \to \infty$. This is related to the running of the coupling in four dimensions.

This solution is a particular case of the general solutions analyzed in [9]. It is a "compactification" with torsion to four dimensions (since the four-dimensional Newton's constant is zero in our case, it is not really a compactification).

The boundary conditions are imposed on seven-dimensional fields; so, it is convenient to work with seven-dimensional gauged supergravity [10].

The form of the supersymmetry variations in seven-dimensional string frame

$$\begin{aligned} \delta\lambda &= \not{D}\phi\epsilon - \frac{i\sqrt{N}}{4}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon + \frac{1}{\sqrt{N}}\epsilon, \\ \delta\chi_\mu &= (D_\mu + iA_\mu)\epsilon - \frac{i\sqrt{N}}{2}F_{\mu\nu}\Gamma^\nu\epsilon. \end{aligned} \tag{1}$$

With the gauge fields $A_\mu = \frac{1}{2}\sigma^a A^a_\mu$, $F = \frac{1}{2}F^a\sigma^a$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$, where σ^a are the Pauli matrices. The solution which correctly describes the UV (or large ρ) in string frame is

$$\begin{aligned} ds^2_{\rm str} &= dx_4^2 + N[d\rho^2 + e^{2g(\rho)}(d\theta^2 + \sin^2\theta d\varphi^2)], \\ e^{2g} &= \rho, \qquad e^{2\phi} = e^{2\phi_0 - 2\rho}\sqrt{\rho}, \qquad A^3 = \cos\theta d\varphi. \end{aligned} \tag{2}$$

We see that this ansatz respects the boundary conditions that we want to impose at infinity.

This metric (2) is singular at $\rho = 0$ and the singularity is of a bad type according to the criteria in [5,11]. To resolve the singularity we consider the symmetries of this solution. The metric we found still has the $U(1)_R$ symmetry; these are U(1) charge rotations in the σ^3 directions in this seven-dimensional description. We expect, however, that this symmetry should be broken by the choice of vacuum in the four-dimensional gauge theory. Naively we expect that the solution should be such that the S^2 should shrink to zero. A similar effect (actually, the opposite) was found in the topological string theory/Chern-Simons correspondence in [12]. (This point was emphasized to us by C. Vafa.) In our case we cannot shrink the S^2 to zero because there is a nontrivial U(1) flux through it. Actually, this problem is completely analogous to a magnetic monopole in SU(2) theory vs the Dirac monopole. So we should look for the solution analogous to the SU(2) monopole, which will have A^1, A^2 nonvanishing. These fields are charged under $U(1)_R$ and will thus break the U(1) symmetry. Fortunately this solution was found in [6] for a four-dimensional gauged supergravity; that has in string frame the same supersymmetry transformation equations. So we can simply read off their solution [6]:

$$\begin{aligned} A &= \frac{1}{2}[\sigma^1 a(\rho)d\theta + \sigma^2 a(\rho)\sin\theta d\varphi + \sigma^3\cos\theta d\varphi], \\ a(\rho) &= \frac{2\rho}{\sinh 2\rho}, \\ e^{2g} &= \rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \\ e^{2\phi} &= e^{2\phi_0}\frac{2e^g}{\sinh 2\rho}. \end{aligned} \tag{3}$$

We see that for large ρ these functions go as $e^{2g} \sim \rho$, $a \sim o(e^{-2\rho})$ and the dilaton also has the same behavior as in the previous U(1) solution. This implies that the solution has the proper UV behavior. At the origin $\rho = 0$ the metric goes as $e^{2g} \sim \rho^2$ so that the metric is nonsingular. It is also easy to check that A is pure gauge at the origin.

Now that we have found the seven-dimensional solution it is possible to lift it up to ten dimensions using the formulas in [7,8]. In order to write the solution it is useful to choose Euler angles on the sphere S^3 and define the left invariant one forms by viewing the sphere as the SU(2) group

$$\begin{aligned} g &= e^{i\psi\sigma^3/2}e^{i\tilde\theta\sigma^1/2}e^{i\phi\sigma^3/2}, \\ \frac{i}{2}w^a\sigma^a &= dgg^{-1}, \\ w^1 + iw^2 &= e^{-i\psi}(d\tilde\theta + i\sin\tilde\theta d\phi), \\ w^3 &= d\psi + \cos\tilde\theta d\phi. \end{aligned} \tag{4}$$

Using the uplifting formulas in [7,8] the ten-dimensional solution is

$$\begin{aligned} ds^2_{\rm str} &= dx_4^2 + N\Big[d\rho^2 + e^{2g(\rho)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\sum_a (w^a - A^a)^2\Big], \qquad e^{2\phi} = e^{2\phi_0}\frac{2e^{g(\rho)}}{\sinh 2\rho}, \\ H^{\rm NS} &= N\Big[-\frac{1}{4}(w^1 - A^1)\wedge(w^2 - A^2)\wedge(w^3 - A^3) + \frac{1}{4}\sum_a F^a\wedge(w^a - A^a)\Big]. \end{aligned} \tag{5}$$

We see that geometrically the resolution of the singularity is the same as that in [13]. If we wrap branes on the S^2 of a resolved conifold the twisted field theory on the brane is precisely what we had above and the resolution is that the S^2 shrinks and the S^3 stays with finite size. In fact, our solution is similar to the solution considered in [13] except that we have only fractional branes and no regular branes.

The fate of the U(1) R-symmetry and the N vacua.—Let us understand why the U(1) symmetry of the solution at infinity is broken to Z_{2N}. In the coordinates we have chosen this U(1) symmetry corresponds to shifting $\psi \to \psi + \epsilon$, with $\psi = \psi + 4\pi$. This symmetry is broken by world sheet instantons, which are the strings of the little string theory wrapping an S^2 inside S^3. More precisely, if we parametrize this S^3 by the coordinates θ, φ of (5) we also have to set $\tilde\theta = \theta$, $\phi = \varphi$, $\psi = \text{const}$. It is possible to have a world sheet with constant ψ thanks to the gauge field A^3, since what appears in the metric is $d\psi + \cos\tilde{\theta} d\phi - \cos\theta d\varphi$. In other words, the coordinate ψ is trivially fibered over the world sheet so that we can pick a configuration with constant ψ. There will be a flux of the B field over this sphere. We can see that, for large ρ,

$$\frac{1}{2\pi}\int_{\psi_2} B - \frac{1}{2\pi}\int_{\psi_1} B = \frac{1}{2\pi}\int H\, d\psi\, d\theta\, d\varphi = -N(\psi_2 - \psi_1)\,. \qquad (6)$$

So this flux goes as $\frac{1}{2\pi}\int_\psi B = b - N\psi$. This flux is the phase that appears in the world sheet instanton calculation. This should be identified with the phase that appears in the field theory instanton calculation, which is the field theory $\theta_{\rm FT}$ angle. We see here that, as we perform a shift in ψ, the phase changes. This implies that the U(1) symmetry is anomalous. We see that $\theta_{\rm FT}$ is not changed if we do rotations by $\psi \to \psi + \frac{2\pi n}{N}$, with $0 \leq n < 2N$. This is precisely the surviving Z_{2N} symmetry in the UV. This symmetry is broken to Z_2 by the solution (5). The surviving Z_2 is just $\psi \to \psi + 2\pi$ which does not change the solution (5).

We should now explain why we have precisely N solutions, or N vacua, for each value of $\theta_{\rm FT}$. First we notice that the world sheet that we were talking about around (6) is contractible in the full geometry. In order to see this we can bring the sphere close to $\rho = 0$ in the geometry (5) and then perform a gauge transformation. After this, the world sheet is wrapped on the two sphere that collapses to zero. If the geometry is to be smooth the flux on the collapsing spherical world sheet better be a multiple of 2π. If we choose a world sheet wrapping S^2 at $\psi = 0$, the flux at the origin is the same as the flux at infinity which in turn is equal to $\theta_{\rm FT}$. So the solution (5) is a good solution only for $\theta_{\rm FT} = 0$. To generate the other solutions we have to rotate the gauge fields by a U(1) transformation $A' = e^{i\psi_0\sigma^3/2}Ae^{-i\psi_0\sigma^2/2}$. This does not change the gauge fields at infinity, so it does not modify the solution in the UV. We can see that the world sheet wrapping near $\rho = \infty$ at the angle ψ_0 can be contracted to the origin with no change in flux. But the flux of this world sheet is $\theta_{\rm FT} - N\psi_0$ and that should be a multiple of 2π so we see that we have N different solutions corresponding to $\psi_0 = \frac{\theta_{\rm FT}}{N} + \frac{2\pi n}{N}$ with $0 \leq n < N$. Thus, from the purely metric point of view, all the solutions with arbitrary values of ψ_0 are nonsingular, but once we consider the B fields we see that only N of the solutions are nonsingular.

A domain wall separating two vacua is localized near $\rho \sim 0$; when we cross the domain wall, we get two different solutions with different values of ψ_0 and we get a change in the flux of B over the contractible sphere by k units if $\Delta\psi_0 = \frac{2\pi k}{N}$. This implies that the gravity dual of the domain wall should be k NS 5-branes wrapping S^3.

It is also possible to see how we can make N of those 5-branes disappear. This is easier to see from the seven-dimensional point of view. The seven-dimensional theory in the variables of [10] has a three form potential. The 5-branes wrapped on S^3 are electrically charged under this three form potential. In the seven-dimensional Lagrangian there is a coupling of the form $iN\int A_3 \wedge {\rm Tr}(F \wedge F)$, where F is the field strength for the ${\rm SU}(2)_R$ gauge fields. So we see that if we have N 5-branes we can replace them by an instanton of the SU(2) gauge field and then expanding the instanton to infinite size we see that this kind of domain wall can disappear. This effect is of course familiar in the heterotic string context where we can transform an NS 5-brane into an instanton in the gauge group [14]. In that case one 5-brane was the same as one instanton.

Towards the pure $\mathcal{N} = 1$ theory.—If we intend to decouple the four-dimensional theory we will have to take a limit where we go to scales much lower than the little string mass scale. As shown in [15] we need to S dualize the gravity solution and switch to a D5-brane description.

The S-dual metric to that in (5) is

$$ds^2_{\rm str} = e^{\phi_D}\left[dx_4^2 + N\left(d\rho^2 + e^{2g(\rho)}d\Omega_2^2 + \frac{1}{4}\sum_a (w^a - A^a)^2\right)\right], \qquad (7)$$

$$e^{2\phi_D} = e^{2\phi_{D,0}}\frac{\sinh 2\rho}{2e^{g(\rho)}}$$

and the NS H field becomes a RR H field. Everything that we said in the previous section about world sheet instantons translates into D-string instantons.

In this description an external quark is a fundamental string that comes in from infinity. When we have a quark-antiquark pair and we separate them by a large distance we see that we find a finite string tension from the point of view of the four-dimensional theory equal to $T_s = \frac{e^{\phi_{D,0}}}{2\pi\alpha'}$. The masses of glueballs and Kaluza-Klein states on the spheres are, in the supergravity approximation, $M^2_{\rm glueballs} \sim M^2_{\rm KK} \sim \frac{1}{N\alpha'}$. Finally the tension of a domain wall interpolating between the nth and $n + 1$th vacua, which is now a D5-brane, is $T_{\rm wall} \sim N^{3/2}e^{2\phi_{D,0}}$. Fundamental strings can end on these domain walls [16]. The baryon vertex is a D3-brane wrapped on S^3. A magnetic monopole source is a D3-brane wrapping the sphere that the world sheet instantons were wrapping in the previous section and extending in the radial and time directions. They are screened because, since the sphere is contractible, each member of a monopole-antimonopole pair can be wrapped in the three-dimensional space parametrized by ρ and the contractible sphere.

We see that in order to decouple the scale of the string tension from the scale of the KK states we need $e^{\phi_{D,0}}N \ll 1$. This goes beyond the gravity approximation, which requires $e^{\phi_{D,0}}N \gg 1$, but it seems that we could still use this metric to formulate a string theory. This string theory should be such that it essentially has no excitations on S^2 or S^3. This is plausible since the sizes of those spheres are smaller than the string scale. Presumably we should be able to replace the six-dimensional part of the geometry by a Liouville-like theory. In fact, since this geometry is similar to the near conifold geometry this sounds plausible. For the near conifold geometry it was suggested in [17] that the sigma model can be replaced, for some calculations, by the $c = 1$ (super) Liouville theory. It would be nice to understand the mapping to a Liouville-like theory in the case that we have RR fields. A nice feature of this RR sigma model is that it seems possible to choose the light cone gauge. In AdS it is hard to choose the light cone gauge, because in Poincaré coordinates we have a horizon. In this case there is no horizon and the light cone theory should be better defined. In the purely four-dimensional theory we do not expect to have any dimensionless parameter. In our case we have a dimensionless parameter which is ϕ_0; this parameter is related to the ratio of the QCD string tension (or mass scale) and the six-dimensional gauge coupling, or six-dimensional scale of the little string theory. Presumably once we exchange the spheres by a Liouville theory we would find that the string coupling is fixed in the IR and of order $1/N$.

Another related point is the precise coefficient for the beta function. In the 5-brane theory it is natural to define the scale as g_{00} in D-string metric, since that will be the energy of a massive string mode sitting at position ρ. This gives a relation between the scale in the field theory and the position ρ of the form $\mu \sim e^{\rho/2}$. When we look at the definition of the four-dimensional string coupling we see that $1/(g_4^2 N) \sim \log \mu/\Lambda_{\mathrm{QCD}}$. But the coefficient is not the correct one. It is interesting that if we go to the five-dimensional Einstein frame metric and we define the scale as $\mu^2 \sim g_{00}^{5,E}$, then we get precisely the right β function with the right numerical coefficient [18]. We could not find any precise reason for choosing this UV/IR relation. In order to determine the precise relation it seems that we should know the precise string theory and sigma model.

In summary, this solution seems to provide a starting point for constructing the large N limit of pure $\mathcal{N} = 1$ Yang-Mills. We expect that the S^3 and S^2 would disappear from the sigma model, leaving only the radial direction, and probably also an angular direction, representing the U(1) symmetry. The final picture would have the flavor of that in [19], but it seems crucial to have RR fields in order to generate a warp factor in string frame.

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