Local Low Dimensionality of Atmospheric Dynamics

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A statistic, the BV (bred vector) dimension, is introduced to measure the effective local finite-time dimensionality of a spatiotemporally chaotic system. It is shown that the Earth's atmosphere often has low BV dimension, and the implications for improving weather forecasting are discussed.

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From the dynamical systems point of view, the behavior of the Earth's atmosphere is extremely high dimensional (e.g., a realistic atmospheric model based on a modal expansion would necessarily include many modes). In spite of the atmosphere's high dimensionality, in this Letter we demonstrate that, in a suitable sense, the *local finite-time* atmospheric dynamics is often low dimensional. Furthermore, as we discuss at the end of this Letter, we believe that this finding has important implications for weather forecasting. More generally, this behavior may be common to other physical spatiotemporally chaotic systems, and these systems may also be amenable to the type of analysis that we introduce for the atmosphere.

The study that we report in this Letter is based on data from numerical weather forecasts posted at regular intervals on the Internet by the United States National Weather Service (NWS). The data provide a unique opportunity to study a state-of-the-art model and a real complex physical system that are closely tied together by data assimilation. Here, the phrase *data assimilation* refers to the process by which the representation of the atmosphere in the computer model is periodically adjusted to attempt to make it consistent with current physical measurements of the atmospheric state.

On 7 December 1992 the NWS implemented operational *ensemble forecasts* [1]. At regular time intervals several perturbations to the model atmospheric state are created. The original atmospheric state (referred to below as the *main solution*) and the ensemble of perturbed states are evolved forward in time by the model to create an ensemble of forecasts.

The difference between the main solution and a perturbed solution is similar to a Lyapunov vector in the familiar calculation of the evolution of differential displacements from chaotic trajectories [2,3]. A difference here is that for the NWS computation, the perturbations, although small, are not infinitesimal. The individual perturbations obtained from the ensemble forecasts are called the *bred vectors* (BV) [4]. If the perturbations were infinitesimal (rather than finite), then, in the limit of infinite time evolution, the bred vectors would point in the direction of dominant growth, and the corresponding exponential growth rate of their magnitudes would be the largest Lyapunov exponent.

For the analysis presented in this Letter, we used ensembles consisting of five perturbed forecasts [3,5]. The ensemble forecasts are made available on the Internet every 24 h by the NWS and give the forecasts at 12-h intervals spanning 8 days [6]. In this study we focus on the wind vector field at the height where the pressure is 500 mbar (approximately 5 km in altitude).

We now describe how these data can be analyzed to obtain a useful local measure of the dimensionality of the atmospheric dynamics. We consider square regions of roughly 1100 km $\times$ 1100 km, choosing a grid point in the center of the region plus 24 uniformly distributed neighbors. The north-south and east-west wind components of a bred vector at the 500 mbar pressure level at each of the 25 points in such a region form a 50-dimensional column vector which we refer to as a *local bred vector.* We normalize each local bred vector in two stages: first we scale the north-south velocities so that their mean squared value is the same as for the east-west velocities. Then we normalize the full 50-dimensional vector to have magnitude one.

If there are k local bred vectors ($k = 5$ in our case), the issue we want to address is the degree of linear independence of these k local bred vectors. That is, we want to determine the effective dimensionality of the subspace spanned by the local bred vectors. To do this we use principal component analysis (PCA) [7] (also known as empirical orthogonal functions). The underlying idea is to find the lowest dimensional subspace that, in a least squares sense, optimally represents the majority of the data.

The k local bred column vectors form a $50 \times k$ matrix, B. The $k \times k$ covariance matrix of B is $C = B^T B$, where B^T is the transpose of B. Since the covariance matrix is non-negative definite and symmetric, its k eigenvalues λ_i are non-negative ($\lambda_i \geq 0$), and its eigenvectors, after multiplying by B and normalizing, form an orthonormal set of vectors v_i which span the column space of B. We order the eigenvalues by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The singular values of B are $\sigma_i = \sqrt{\lambda_i}$. The eigenvalues λ_i are a measure of the extent to which the k column unit vectors making up B point in the direction v_i, and each $\sigma_i^2 = \lambda_i$ is said to represent the amount of variance in the set of the k unit vectors that is accounted for by v_i. By identifying how many of the vectors of v_i represent most of the variance of B we can identify an effective dimension spanned by the

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k local bred vectors. For example, if two out of five singular values are zero, then the subspace spanned by the k local bred vectors is three dimensional. However, if some of the singular values are nonzero but small, the issue becomes more difficult. One option is to use thresholding (as is often done when trying to isolate the dominant modes in PCA), but there is difficulty in determining the "best" value of the threshold. Instead we opt to define the following statistic on the singular values which we call the BV dimension (bred vector dimension):

$$\psi(\sigma_1, \sigma_2, \ldots, \sigma_k) = \frac{(\sum_{i=1}^{k} \sigma_i)^2}{\sum_{i=1}^{k} \sigma_i^2}. \tag{1}$$

Since each column of B (the local bred vectors) is of unit length, $\sum_{i=1}^{k} \sigma_i^2 = k$ [7]. As examples of the statistic, we describe several cases with $k = 5$. If the k local bred vectors comprising B were all the same, then the singular values would be $\sqrt{5}, 0, 0, 0, 0$. This would yield a statistic of $\psi(\sqrt{5}, 0, 0, 0, 0) = 1$. If the local bred vectors were equally distributed between two vectors v_1 and v_2, in the sense that each one accounts for half the variance, then the singular values would be $\sqrt{5/2}, \sqrt{5/2}, 0, 0, 0$, and our statistic would yield $\psi(\sqrt{5/2}, \sqrt{5/2}, 0, 0, 0) = 2$. If the local bred vectors again lie in the two-dimensional subspace spanned by v_1 and v_2, but the two are not equally represented, then this could give $2 \geq \psi \geq 1$ [e.g., we might have singular values such as $\sqrt{4.75}, 1/2, 0, 0, 0$ which would yield $\psi(\sqrt{4.75}, 1/2, 0, 0, 0) \approx 1.4$]. While the dimension of the space spanned by the local bred vectors is 2, our statistic gives an intermediate value reflecting the degree of dominance of one direction over the other. In general our statistic returns a real value between 1 and k. Note that while small perturbations due to noise or numerical error will typically cause the dimensionality of the space spanned by the k local bred vectors to be k, the effective dimension ψ may be substantially lower and is robust to small changes in the σ_i due to noise or numerical error.

In Fig. 1 we show this analysis applied to a 36-h forecast. The BV dimension was calculated at each spatial point on the grid and colored red for lower values and blue for larger values. A large region of relatively low dimensionality (BV dimension less than 2.5) is evident over North America (red-yellow). This indicates that in this region, the local bred vectors effectively span a space of substantially lower dimension than that of the full space. We also find that there is a well defined vertical structure (from 850 to 250 mbar) in the atmospheric column of regions with low BV dimensions.

In order to check that the low BV-dimension regions are statistically significant and not due to random fluctuations, we apply the BV dimension to surrogate data consisting of bred vectors chosen from different days which are substantially far apart (this removes temporal correlations). When this is done we observe no regions with low BV dimension. An example of such a surrogate is shown in Fig. 2. In this surrogate there are no values of BV dimension less than 3.

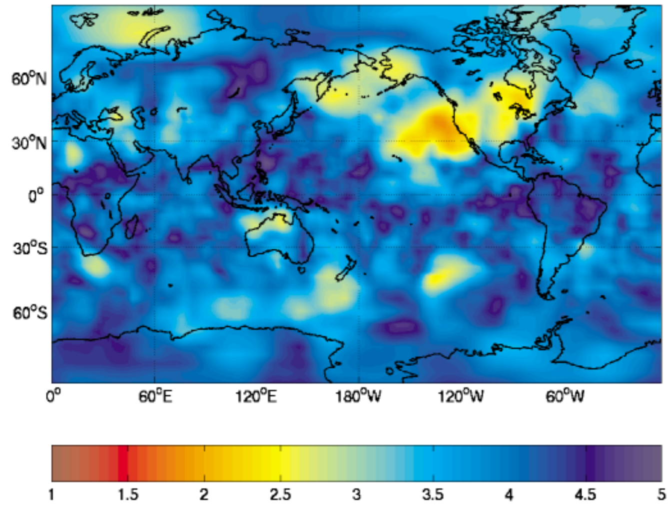

FIG. 1 (color). An example of the spatial variation of BV dimension from a 36-h forecast from 5 March 2000. The colors represent the effective subspace spanned by the local bred vectors (BV dimension), with red indicating lower values and blue indicating higher values.

Low values of BV dimension, over time, tend to follow fairly defined tracks in our data set moving from west to east. The average location of regions with low BV dimension for the time period of 10 February 2000 to 30 July 2000 is exhibited in Fig. 3 [8]. For the time period studied, much of the low BV-dimension areas were concentrated over North America. We suspect that the disparity between northern and southern hemispheres is due to differences in the structure of storm tracks between the hemispheres.

The regions of low BV dimension have potentially important implications for weather forecasting. A major

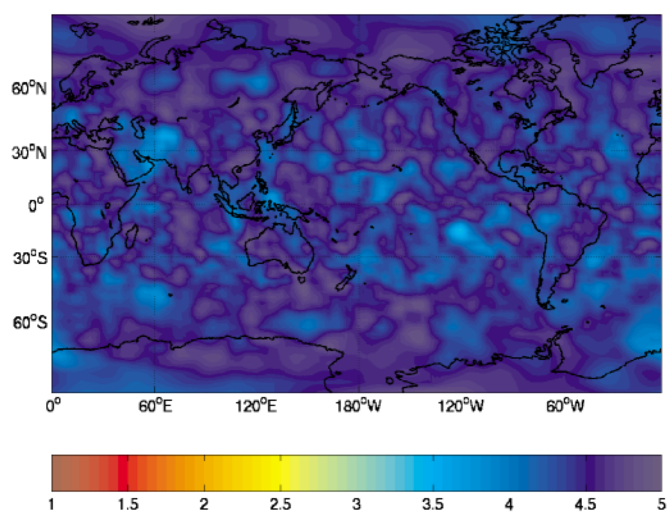

FIG. 2 (color). Regions of low BV dimension are not due to random correlations. Above is an example of the BV dimensions computed on a surrogate (described in the text) for which temporal correlations have been removed. Note the lack of any regions with BV dimension less than 3.

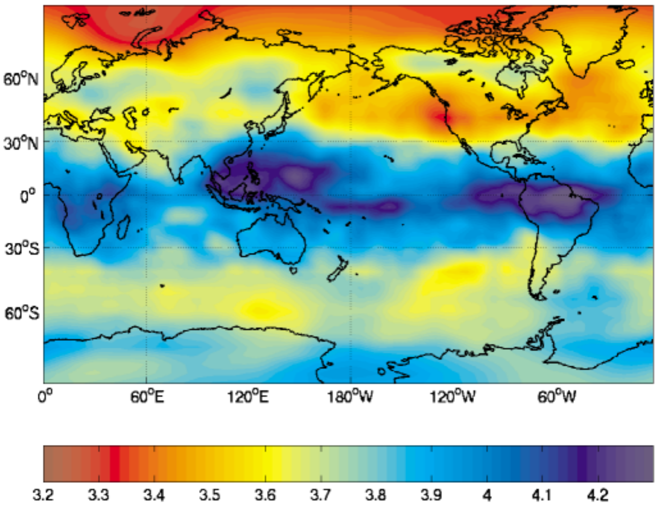

FIG. 3 (color). Average locations of regions with low BV dimensions are shown through the pointwise time average of the BV dimension calculated from ensemble forecasts every 12 h from 10 February 2000 to 30 July 2000. Red (blue) depicts regions in which the BV dimension tends to be low (high).

effort in forecasting is devoted to the process of data assimilation, whereby measurements are used to adjust the representation of the atmosphere in the computer model so as to make it more closely conform to the actual current state of the atmosphere. The data assimilation is the primary means to establish the initial conditions for the forecast computations. Clearly, this is key to weather forecasting, since the quality of the forecasts depends on the accuracy of the initial conditions. On the other hand, the full atmospheric state is not measured, but rather only field variables at a finite number of measurement locations. Furthermore, these measurements have errors. At any given time t_0, there is inevitably a discrepancy $\vec{\Delta}(t_0)$ between the true atmospheric state and its representation in the computer model. Now consider a later time $t_1 > t_0$, and suppose that in a region of interest there is a low BV dimension at time t_1. This implies that any local discrepancy $\vec{\Delta}(t_1)$ between the true state and its representation in the computer model lies predominantly in the "unstable subspace," the space spanned by the few vectors that contribute most strongly to the low BV dimension [9]. We conjecture that in many cases this information can yield a substantial improvement in forecasting. In particular, the implication is that the data assimilation algorithm should correct the computer model state by moving it closer to the observations along the direction of the unstable subspace since that is where the true state most likely lies [10]. Current data assimilation techniques (e.g., that used by the NWS) do not take this into account. Furthermore, we suggest the need to study the results if the number of forecasts in the ensemble is substantially increased. This would give more bred vectors. For example, with 100 bred vectors it might result that a very large fraction of the time the local BV dimension is six or less. If so, then our remarks above on data assimilation would apply to this larger (but still relatively low) dimensional case. (With the current smaller ensemble of five bred vectors available from the NWS, such a question cannot be resolved.)

In conclusion, our main result is a means of establishing local low-dimensional behavior (the BV dimension) of a spatiotemporally chaotic system, and the demonstration that the Earth's atmosphere exhibits locally low-dimensional behavior [11]. Furthermore, we conjecture that this finding can be used as a basis for improving data assimilation techniques for weather forecasting.

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[2] K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems* (Springer, New York, 1997); E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993).
[3] Z. Toth and E. Kalnay, Mon. Weather Rev. **125**, 3297 (1997).
[4] Because the purpose of the ensemble forecast is primarily operational, the NWS periodically rescales the bred vectors in a spatially nonuniform manner to match the amplitude of observational errors in different regions. As a result, growth rates of the perturbations from the main solution cannot be obtained from the data reported by the NWS.
[5] Ten bred vectors are made available, but five of them are the result of subtracting the perturbations from the main solution; hence we average the pairs of the perturbations to give five bred vectors.
[6] The NWS global model output requires almost a gigabyte of data; hence only a few key fields are posted.
[7] J. T. Scheick, *Linear Algebra with Applications* (McGraw-Hill, New York, 1997).
[8] The locations of regions with low BV dimension show a striking resemblance pattern of low-dimensional atmospheric instabilities identified by D. J. Patil, B. R. Hunt, and J. A. Carton (to be published).
[9] D. J. Patil, B. R. Hunt, E. Kalnay, E. Ott, and J. Yorke (to be published).
[10] We will discuss assimilation techniques based on this idea in a future publication. See also E. Kalnay and Z. Toth,

in Proceedings of the Tenth Conference on Numerical Weather Prediction, 1994.

[11] We emphasize that the type of low dimensionality that we claim is very different from that of the previous works on the dynamics of atmosphere. In particular, our results are with respect to the *local finite-time* dynamics. In the past there have been claims for global low-dimensional atmospheric dynamics that are based on the application of embedding theorems to time series data [e.g., C. Nicolis and G. Nicolis, Nature (London) **311**, 529 (1986)]. However, these results have been strongly questioned [e.g., P. Grassberger, Nature (London) **323**, 609 (1986)].