Freak Waves in Random Oceanic Sea States

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Freak waves are very large, rare events in a random ocean wave train. Here we study their generation in a random sea state characterized by the Joint North Sea Wave Project spectrum. We assume, to cubic order in nonlinearity, that the wave dynamics are governed by the nonlinear Schrödinger (NLS) equation. We show from extensive numerical simulations of the NLS equation how freak waves in a random sea state are more likely to occur for large values of the Phillips parameter α and the enhancement coefficient γ. Comparison with linear simulations is also reported.

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Freak waves are extraordinarily large water waves whose heights exceed by a factor of 2.2 the significant wave height, H_s, of a measured wave train [1]. The mechanism of freak wave generation has become an issue of principal interest due to their potentially devastating effects on offshore structures and ships. In addition to the formation of such waves in the presence of strong currents [2] or as a result of a simple chance superposition of Fourier modes with coherent phases, it has recently been established that the nonlinear Schrödinger (NLS) equation can describe many of the features of the dynamics of freak waves which are found to arise as a result of the nonlinear self-focusing phenomena [3–5]. The self-focusing effect arises from the Benjamin-Feir instability [6]: a monochromatic wave of amplitude a_0 and wave number k_0 perturbed on a wavelength $L = 2\pi/\Delta k$, is unstable whenever $\Delta k/(k_0 \varepsilon) < 2\sqrt{2}$, where ε is the steepness of the carrier wave defined as $\varepsilon = k_0 a_0$. The instability causes a local exponential growth in the amplitude of the wave train. This result is established from a linear stability analysis of the NLS equation [7] and has been confirmed experimentally (see, for example, [8] and references therein) and from numerical simulations of the fully nonlinear water wave equations [5]. Even though the above results are well understood and robust from a physical [7] and mathematical [9,10] point of view, it is still unclear how freak waves are generated via the Benjamin-Feir instability in more realistic oceanic conditions.

In this Letter, our attention is focused on freak wave generation in numerical simulations of the NLS equation where we assume initial conditions typical of oceanic sea states described by the Joint North Sea Wave Project (JONSWAP) power spectrum (see, e.g., [11]):

$$P(f) = \frac{\alpha}{f^5} \exp\left[-\frac{5}{4}\left(\frac{f_0}{f}\right)^4\right] \gamma^{\exp -[-[(f-f_0)^2]/(2\sigma_0^2 f_0^2)]}, \tag{1}$$

where $\sigma_0 = 0.07$ if $f \le f_0$ and $\sigma_0 = 0.09$ if $f > f_0$. Our use of the JONSWAP formula is based upon the established result that developing storm dynamics are governed by this spectrum for a range of the parameters [11]. Here f_0 is the dominant frequency, γ is the "enhancement" coefficient, and α is the Phillips parameter. For $\gamma = 1$ and $\alpha = 0.0081$ the spectrum reduces to that of Pierson and Moskowitz [11]. As γ increases, the spectrum becomes higher and narrower around the spectral peak. In Fig. 1 we show the JONSWAP spectrum for different values of γ ($\gamma = 1, 5, 10$) for $f_0 = 0.1$ Hz and $\alpha = 0.0081$.

The major finding we would like to discuss herein is that as γ and α grow, nonlinearity becomes more important and the probability of the formation of freak waves increases. Our results have been achieved by considering the NLS equation as the *simplest* nonlinear evolution equation for describing deep-water wave trains. We have performed numerical simulations using the JONSWAP spectrum to determine the initial conditions. Since the analytical form of the spectrum is given as a function of frequency, analysis is carried out by considering the so-called timelike NLS equation (TNLS) (for the use of timelike equations in water waves, see, e.g., [12–14]), which describes the evolution of the complex envelope A in deep water waves:

$$A_x + i\left(\frac{\Delta\omega}{\omega_0}\right)^2 A_{tt} + i\varepsilon^2 |A|^2 A = 0, \tag{2}$$

where dimensional quantities denoted with primes have been scaled according to $A = a_0 A'$, $x = x'/k_0$, and $t = t'/\Delta\omega$ with $1/\Delta\omega$ a characteristic time scale of the envelope which corresponds to the width of the frequency

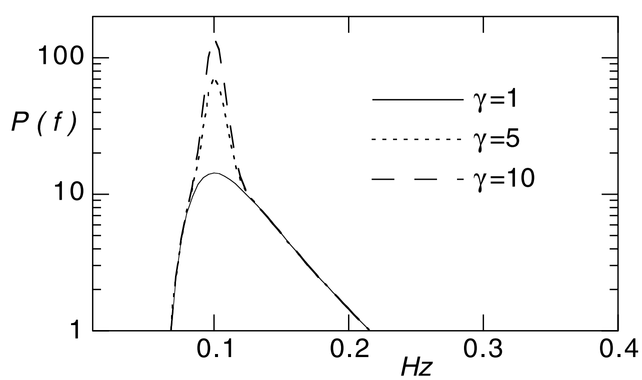

FIG. 1. The JONSWAP spectrum for $\gamma = 1$ (solid line), $\gamma = 5$ (dotted line), and $\gamma = 10$ (dashed line); with $f_0 = 0.1$ Hz and $\alpha = 0.0081$.

spectrum. Equation (2) solves a *boundary value problem:* given the temporal evolution $A(0,t)$ at some location $x = 0$, Eq. (2) determines the wave motion over all space, $A(x,t)$.

At this point, it is instructive to introduce a parameter that estimates the influence of the nonlinearity in deep water waves. This parameter, which is a kind of "Ursell number" [14], can be obtained as the ratio of the nonlinear and dispersive terms in the TNLS equation:

$$Ur = \left(\frac{\varepsilon}{\Delta\omega/\omega_0}\right)^2. \tag{3}$$

When $Ur \ll 1$ waves are essentially linear; for $Ur \geq 1$, the dynamics become nonlinear and the evolution of the wave train is likely dominated by envelope solitons or unstable mode solutions such as those studied by Yuen and co-workers [7].

Many aspects of the importance of the nonlinearity can be addressed by computing Ur from the spectrum (1). In Fig. 2 we show the Ursell number as a function of the parameter γ for $\alpha = 0.0081$ and $\alpha = 0.0162$. In the construction of the plot an estimation of ε and $\Delta\omega/\omega_0$ needs to be given. $\Delta\omega$ has been estimated as the half width at half maximum of the spectrum and the steepness as $\varepsilon = k_0 H_s/2$. From the plot it is evident that for the Pierson-Moskowitz spectrum ($\gamma = 1$) the Ursell number is quite small: this indicates that dispersion dominates nonlinearity. It has to be pointed out that for small values of γ ($\gamma = 1, 2$) the spectrum is not narrow banded; as γ increases the spectrum becomes narrower ($\Delta\omega/\omega_0 \simeq 0.2$ or less), suggesting that the NLS equation is more appropriate. For large values of γ the mean steepness increases; for $\gamma = 8$, $\alpha = 0.01$ the steepness is equal to 0.16, therefore the equation is no longer valid and higher order terms in steepness are required. In Fig. 2 we have placed vertical lines at $\gamma = 2.5$ and $\gamma = 8$ to indicate the region in which the NLS equation is applicable.

When the spectral width becomes large, one expects results which are somewhat out of the range of applicability of the NLS equation. As pointed out in a number of papers [15,16] the main defect in the NLS equation concerning the narrow-band approximation, arises from the fact that linear dispersion is not at high enough order. Reference [16] proposes an equation that includes all the terms in the linear dispersion relation. The equation [Eq. (1) in their paper], which is basically the NLS equation with the full linear dispersion relation of the primitive equations, "reproduces exactly the conditions for nonlinear four-wave resonance even for bandwidth greater than unity." In the linear limit the equation is exact. In our numerical simulations with the Pierson Moskowitz spectrum ($\gamma = 1, \alpha = 0.0081$) we have used both the NLS equation and its modified form [Eq. (1) in [16]]. In this specific case the two equations give basically the same results: nonlinearities are weak (Ursell number $= 0.03$; see Fig. 2) and the dynamics are basically linear; the correction in the linear dispersion relation does not essentially alter the value of the maximum simulated wave amplitudes. The results of these tests have convinced us that, for the important range $\gamma = 2.5-8$, the simpler NLS equation is a valid approach for studying many of the properties of rogue waves.

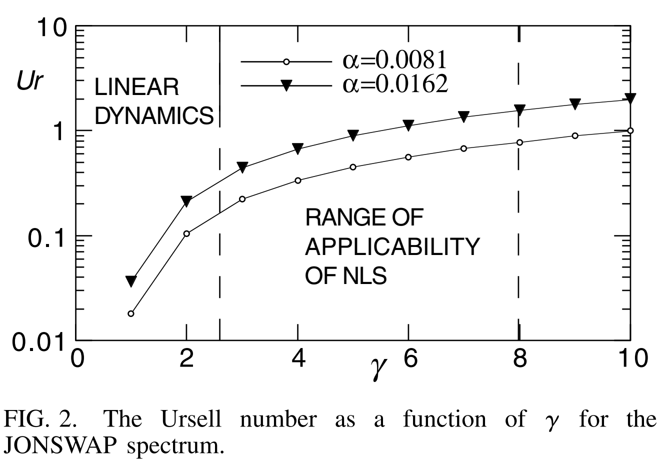

FIG. 2. The Ursell number as a function of γ for the JONSWAP spectrum.

The influence of the parameter α consists in increasing the energy content of the time series and therefore, as α increases, the wave amplitude and, consequently, the wave steepness also increase. If α doubles, the steepness increases by a factor of $\sqrt{2}$ and the Ursell number by a factor of 2 since the spectral width remains constant. From this analysis we expect that large amplitude freak waves (large with respect to their significant height) are more likely to occur when γ and α are both large.

We now consider numerical simulations of Eq. (2) which have been computed using a standard split-step, pseudospectral Fourier method [12]. Initial conditions for the free surface elevation $\zeta(0,t)$ have been constructed as the following random process [17]:

$$\zeta(0,t) = \sum_{n=1}^{N} C_n \cos(2\pi f_n t - \phi_n), \tag{4}$$

where ϕ_n are uniformly distributed random numbers on the interval $(0, 2\pi)$, and $C_n = \sqrt{2P(f_n)\Delta f_n}$, where $P(f)$ is the JONSWAP spectrum given in (1). For the computational domain considered, it was checked that the shape of the JONSWAP spectrum has been not substantially altered during the evolution of the NLS equation. In Fig. 3 we show an image of smoothed contours of a space-time field of $|A|$ from a numerical simulation of TNLS obtained with $\gamma = 4$. The dominant frequency and the Phillips parameter of the initial wave train were set, respectively, to 0.1 Hz and $\alpha = 0.02$. A large amplitude wave appears in the simulation and in order to better visualize it, in Fig. 4 we show a time series of the free surface $\zeta(t)$ at $x = 1550$ m obtained using the following relation:

$$\zeta(t) = [A(t)e^{i2\pi f_0 t} + \text{c.c.}]/2, \tag{5}$$

where c.c. denotes complex conjugate. A freak wave of about 18.5 m in a random wave train with significant

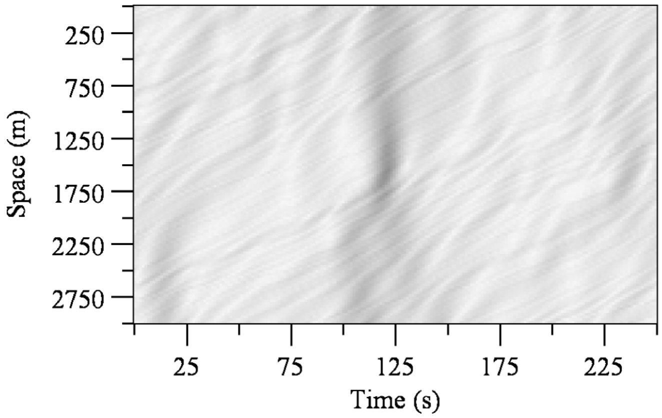

FIG. 3. Nonlinear Schrödinger space-time evolution of a random wave field using the JONSWAP spectrum with $\gamma = 4$, $\alpha = 0.0081$. For details refer to the text. Gray scale ranges from 0 m (white) to 12.8 m (dark).

wave height of $H_s = 6.9$ m is evident at time $t = 140$ s. We point out that the same simulation, not reported here for brevity, with exactly the same initial conditions, has been performed after setting to zero the term $|A|^2A$. No waves fulfilling the freak wave threshold ($H > 2.2H_s$) were found in the domain considered.

In order to give additional quantitative results we have performed more that 300 simulations of the TNLS equation. The simulations have been performed in dimensional units in the following way. An initial time series of 250 s has been computed from the JONSWAP spectrum for different values of α (from $\alpha = 0.0081$ to $\alpha = 0.02$) and γ ($\gamma = 1$, $\gamma = 4$, and $\gamma = 10$). To increase the number of statistical events we made computer runs with 10 different sets of random phases, ϕ_n. The time series were then evolved according to the TNLS for a distance of 10 km, saving the output every 10 m. From an experimental point of view, this approach corresponds to setting 1000 probes along the wave propagation direction (one every 10 m) and measuring each time series for 250 s at a sampling frequency of 2.05 Hz. The significant wave height, H_s, of each realization has been computed; the highest wave, $H_{\max}$, has been found and the ratio $H_{\max}/H_s$ has been determined. In order to verify that, in all the simulations performed, our findings are really a consequence of the nonlinear dynamics, we have also computed exactly the same simulations using the linear version of the TNLS equation. The results are summarized in Figs. 5 and 6. Figure 5 corresponds to $\gamma = 1$. A horizontal line at $H_{\max}/H_s = 2.2$ indicates the threshold that arbitrarily discriminates the height of rogue waves. For the Pierson Moskowitz spectrum ($\gamma = 1$ and $\alpha = 0.0081$) only one realization of the 10 considered shows a "rogue" wave with $H_{\max}/H_s = 2.25$. For higher values of α only a few of the realizations show waves with $H_{\max}/H_s$ slightly greater than 2.2. From the plot it is clear that the effects of nonlinearities are rather small for this case ($\gamma = 1$). Among all the 50 linear simulations performed with $\gamma = 1$, we have encountered a number of large amplitude waves but none exceeds $2.2H_s$. For $\gamma = 4$, see Fig. 6, the physical picture becomes much more interesting: while in the linear simulations there are no freak waves, 50% of the nonlinear simulations performed show at least one freak wave. For larger γ the picture is qualitatively the same and therefore the graphs are not reported. There is clear evidence that increasing γ increases the probability of freak wave occurrences; high values of γ do not, however, guarantee the presence of a giant wave. The local properties of the wave trains are presumably of fundamental importance for understanding the formation of freak waves: it may happen that the Benjamin-Feir instability mechanism is satisfied only in a small temporal portion of the full wave train, giving rise to a local instability and therefore to the formation of a freak wave.

From a physical point of view, we are aware of the fact that the NLS equation overestimates the region of instability and the maximum wave amplitude with respect to

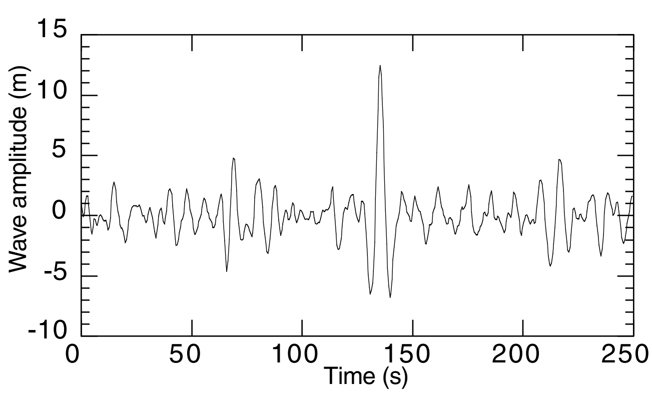

FIG. 4. Free surface elevation $\zeta(t)$ at $x = 1550$ m obtained from Fig. 3.

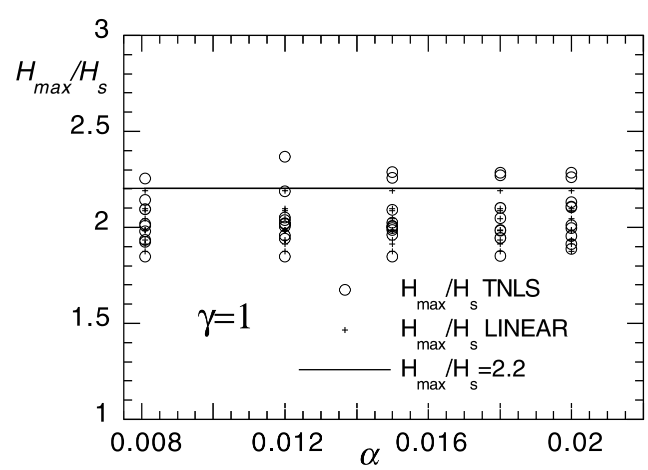

FIG. 5. $H_{\max}/H_s$ as a function of α for $\gamma = 1$. Circles and crosses correspond, respectively, to the nonlinear (TNLS) and linear simulations. For each value of α, ten different realizations corresponding to ten different sets of random numbers have been performed.

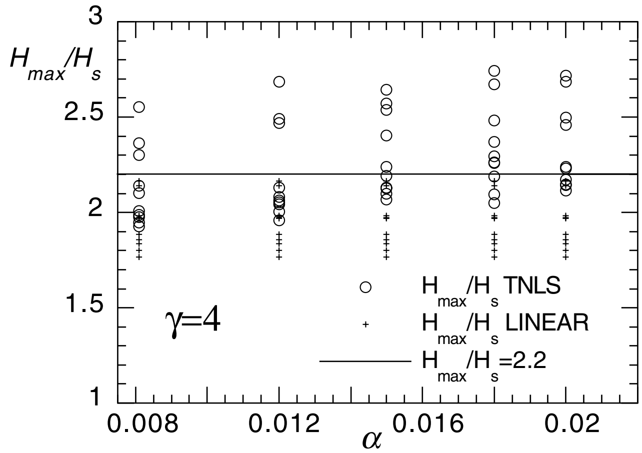

FIG. 6. $H_{\max}/H_s$ as a function of α for $\gamma = 4$. See caption of Fig. 5 for details.

higher order models [18], especially for ε greater than 0.1. Furthermore, it is well known that the NLS equation is formally derived from the Euler equations under the assumption of a narrow-banded process. Nevertheless, in spite of these deficiencies in the NLS equation, we believe that our results provide new important physical insight into the generation of freak waves. Simulations with higher order models [18] or directly with the fully nonlinear equations of motion will be required in order to confirm these results. Wave tank experiments will also be very useful in this regard.

Another issue that has to be taken into account for future work is directional spreading. In a recent paper [19] we have considered simple initial conditions using the NLS equation in 2 + 1 dimensions, and we have found the ubiquitous occurrence of freak waves. Whether the additional directionality in the JONSWAP spectrum changes our statistics is still an open question; at the same time we are confident that our results can apply to the case in which the spectrum is quasi-unidirectional. In particular, as recently suggested [20], the so called "energetic swells," which correspond to the early stage of swell development, still characterized by a highly nonlinear regime (peaked spectra and large values of α are candidates for the occurrence of freak waves).

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