Kinetic Equation with Exact Charge Conservation

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A kinetic master equation for multiplicity distributions is formulated for charged particles which are created or destroyed only in pairs due to the conservation of their Abelian charge. It allows one to study time evolution of the multiplicity distributions in a relativistic many-body system with arbitrary average particle multiplicities. It is shown to reproduce the equilibrium results for both canonical (rare particles) and grand canonical (abundant particles) systems. For canonical systems, the equilibrium multiplicity is much lower and the relaxation time is much shorter than the naive extrapolation from grand canonical results. Implications for chemical equilibration in heavy-ion collisions are also discussed.

I. Introduction.—Relativistic statistical thermodynamics has long been used as a tool to describe particle production in heavy-ion and in high-energy particle collisions [1–3]. Recent analyses have shown that the statistical models can indeed give a satisfactory description of the multiplicities of most hadrons measured in A-A collisions at Alternating Gradient Synchrotron (at BNL) and CERN Super Proton Synchrotron (SPS) energies [4,5]. However, the dynamics of particle equilibration and, in particular, of chemical equilibration is still not well understood [6–8].

Within a statistical kinetic approach, particle production is commonly described using the grand canonical ensemble, where event-averaged multiplicities are controlled by chemical potentials. In this description the net value of a given U(1) charge (e.g., electric, baryon, strangeness, charm, etc.) fluctuates. These fluctuations can be neglected only if the particles carrying the charge in question are abundant. In this case the charge will be conserved on the average. In the opposite limit of rare particle production, conservation laws must be implemented locally on an event-by-event basis [1,2,9,10]; i.e., a canonical ensemble must be used. The kinetic equations for the production of strongly correlated particles were studied in the literature [11,12] with consideration of multiplicity fluctuations. However, no complete solution has been obtained.

The local conservation of quantum numbers in the canonical approach severely reduces the phase space available for particle production [1,2,5,9,10,13–15]. Recently, it has been shown that the canonical statistical model provides a good description of particle yields measured in low-energy heavy-ion [16] and high-energy hadron-nucleus, hadron-hadron, and e^+e^- reactions [5,17,18].

In view of these results it is of particular importance to consider a new approach that rigorously takes into account the multiplicity fluctuations. In this Letter, we formulate for the first time a kinetic theory for the time evolution of particle *distributions* that allows us to investigate the approach to the canonical chemical equilibrium, both the average multiplicity and its fluctuations. We consider the multiplicity distribution of particles that carry the charges corresponding to a U(1) internal symmetry. The kinetic master equation is valid for arbitrary average particle multiplicities. It reproduces the canonical equilibrium solution for rare particle production and, obviously, reduces to the standard grand canonical rate equation for abundant particle production.

II. Rate equation.—In the standard formulation [11,19–21], the rate equation for a binary process $a_1 a_2 \to b_1 b_2$ with $a \neq b$ is described by the following population equation:

$$\frac{d\langle N_{b_1}\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N_{b_1}\rangle\langle N_{b_2}\rangle, \quad (1)$$

where $G \equiv \langle \sigma_G v\rangle$ and $L \equiv \langle \sigma_L v\rangle$ give the momentum-averaged cross sections for the gain process $a_1 a_2 \to b_1 b_2$ and the loss process $b_1 b_2 \to a_1 a_2$, respectively. N_k represents the total number of particles k, and V is the proper volume. Among such processes, a typical example is the kaon production/annihilation via $\pi^+ \pi^- \leftrightarrow K^+ K^-$. The above rate equation, however, cannot be applied to the situation where particle production is rare and is strongly correlated by exact charge conservation.

To account for the correlation between the production/annihilation of particles b_1 and b_2, let us define $P_{i,j}$ as the probability to find a number i of particle b_1 and a number j of particle b_2 in an event. We also denote as P_i the probability to find a number i of particle b in an event. The average number of particles b per event is then defined as

$$\langle N_b\rangle = \sum_{i=0}^{\infty} i P_i. \quad (2)$$

We can now write the following general rate equation for the average particle multiplicity [22]:

$$\frac{d\langle N_{b_1}\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\sum_{i,j} ijP_{i,j}. \qquad (3)$$

Further, let particles b_1 and b_2 carry opposite units of a charge corresponding to a U(1) internal symmetry (strangeness in the case of kaons). Then the U(1) charge neutrality of the system gives $N \equiv N_{b_1} = N_{b_2}$. We have then

$$P_{i,j} = P_i\delta_{ij},$$
$$\sum_{i,j} ijP_{i,j} = \sum_i i^2 P_i \equiv \langle N^2\rangle = \langle N\rangle^2 + \langle\delta N^2\rangle, \qquad (4)$$

where $\langle\delta N^2\rangle$ represents the event-by-event fluctuation of the number of b_1b_2 pairs. Note that we always consider a_1 and a_2 particles abundant (e.g., pions) so that we can neglect the event-by-event fluctuations of their multiplicity and the change of their number due to the considered processes.

Following Eqs. (3) and (4), the general rate equation for the average number of b_1b_2 pairs can be written as

$$\frac{d\langle N\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N^2\rangle. \qquad (5)$$

For abundant production of b_1b_2 pairs, where $\langle N\rangle \gg 1$,

$$\langle N^2\rangle \approx \langle N\rangle^2, \qquad (6)$$

and Eq. (5) obviously reduces to the standard form, i.e.,

$$\frac{d\langle N\rangle}{d\tau} \approx \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N\rangle^2. \qquad (7)$$

However, for rare production of b_1b_2 pairs, where $\langle N\rangle \ll 1$, the rate equations (1) and (7) are no longer valid. In this case, we have $P_1 \gg P_i$ for $i \geq 2$; thus following Eqs. (2) and (4) we can write

$$\langle N^2\rangle \approx \langle N\rangle, \qquad (8)$$

which reduces Eq. (5) to the following form:

$$\frac{d\langle N\rangle}{d\tau} \approx \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N\rangle. \qquad (9)$$

Thus, in the limit $\langle N\rangle \ll 1$, the absorption term depends on the pair number only *linearly,* instead of quadratically for the limit $\langle N\rangle \gg 1$.

III. Equilibrium multiplicity and relaxation time.—To illustrate the differences in the time evolution of particle abundance, we consider the two limiting cases, $\langle N\rangle \gg 1$ and $\langle N\rangle \ll 1$, and present their equilibrium values and relaxation times for the production of b_1b_2 pairs. As an example, we consider a system at fixed temperature and volume and with no initial b_1b_2 pairs, i.e., $\langle N\rangle(\tau = 0) = 0$.

In the limit when $\langle N\rangle \gg 1$, the standard Eq. (7) is valid and has the following well-known solution:

$$\langle N\rangle^{\mathrm{GC}}(\tau) = N_{\mathrm{eq}}^{\mathrm{GC}}\tanh(\tau/\tau_0^{\mathrm{GC}}), \qquad (10)$$

where the equilibrium value for the number of b_1b_2 pairs $N_{\mathrm{eq}}^{\mathrm{GC}}$ and the relaxation time constant τ_0^{GC} are given by

$$N_{\mathrm{eq}}^{\mathrm{GC}} = \sqrt{\epsilon}, \qquad \tau_0^{\mathrm{GC}} = \frac{V}{L\sqrt{\epsilon}}, \qquad (11)$$

respectively, where

$$\epsilon \equiv G\langle N_{a_1}\rangle\langle N_{a_2}\rangle/L. \qquad (12)$$

In the special case where particle momentum distributions are thermal, the gain (G) and loss (L) terms represent just the thermal averages of the production and absorption cross sections. From the definition for the thermal averaged cross section [11] and neglecting quantum statistics [23], we find

$$\frac{G}{L} = \frac{d_{b_1}\alpha_{b_1}^2 K_2(\alpha_{b_1})d_{b_2}\alpha_{b_2}^2 K_2(\alpha_{b_2})}{d_{a_1}\alpha_{a_1}^2 K_2(\alpha_{a_1})d_{a_2}\alpha_{a_2}^2 K_2(\alpha_{a_2})}, \qquad (13)$$

where d_k's denote the degeneracy factors, $\alpha_k \equiv m_k/T$, and K_2 represents the modified Bessel function. The equilibrium value for the number of b_1b_2 pairs in Eq. (11) now reads as

$$N_{\mathrm{eq}}^{\mathrm{GC}} = \frac{d_{b_1}}{2\pi^2}VT^3\alpha_{b_1}^2 K_2(\alpha_{b_1}). \qquad (14)$$

Thus it is described by the grand canonical (GC) result with vanishing chemical potential due to our requirement of the (average) U(1) charge neutrality of the system.

In the opposite limit where $\langle N\rangle \ll 1$, the time evolution is described by Eq. (9), which has the following solution:

$$\langle N\rangle^{\mathrm{C}}(\tau) = N_{\mathrm{eq}}^{\mathrm{C}}(1 - e^{-\tau/\tau_0^{\mathrm{C}}}), \qquad (15)$$

with the equilibrium value and relaxation time given by

$$N_{\mathrm{eq}}^{\mathrm{C}} = \epsilon, \qquad \tau_0^{\mathrm{C}} = \frac{V}{L}. \qquad (16)$$

With a thermal momentum distribution the equilibrium value of b_1b_2 pair multiplicity has the following form:

$$N_{\mathrm{eq}}^{\mathrm{C}} = \left[\frac{d_{b_1}}{2\pi^2}VT^3\alpha_{b_1}^2 K_2(\alpha_{b_1})\right] \cdot \left[\frac{d_{b_2}}{2\pi^2}VT^3\alpha_{b_2}^2 K_2(\alpha_{b_2})\right]. \qquad (17)$$

We note that the above equation is formally equivalent to the product of the multiplicity for particle b_1 and that for particle b_2 in the limit of grand canonical equilibrium as given by Eq. (14). This equation thus demonstrates the locality of the U(1) charge conservation. With each particle b_1, a particle b_2 with the opposite charge is produced in the same event in order to conserve charge locally. This is the result expected from the canonical (C) formulation of conservation laws [10,13,14].

We note that Eq. (17) is just the leading term in the expansion of the canonical result for the multiplicity of particles carrying the U(1) charges. The general expression is known to have the following form [10,13,14]:

$$N_{eq}^{C} = N_{eq}^{GC} \frac{I_1(2N_{eq}^{GC})}{I_0(2N_{eq}^{GC})}, \qquad (18)$$

where N_{eq}^{GC} is given by Eq. (14) and I_i's are the modified Bessel functions.

Comparing Eq. (11) and Eq. (16), we first find that, for $\langle N \rangle \ll 1$, the equilibrium multiplicity in the canonical formulation is much lower than what is expected from the grand canonical result,

$$N_{eq}^{C} = (N_{eq}^{GC})^2 \ll N_{eq}^{GC}. \qquad (19)$$

This shows the importance of the canonical description of charge conservation when the multiplicity of charged particles is small. We also note that the volume dependence in the two cases is different. The particle density in the GC (abundant) limit is independent of V, whereas in the opposite canonical (rare) limit the density scales linearly with V.

Second, the relaxation time for a canonical system is far shorter than what is expected from the grand canonical result,

$$\tau_0^{C} = \tau_0^{GC} N_{eq}^{GC} \ll \tau_0^{GC}, \qquad (20)$$

due to the small number of particles ($N_{eq}^{GC} \ll 1$). For example, the total number of produced kaons in Au + Au collisions at 1 GeV/A is of the order of 0.02. Thus the canonical relaxation time is a factor of 7 shorter than what is expected from the grand canonical formulation.

We note from Eq. (5) that these two limits are essentially determined by the size of $\langle \delta N^2 \rangle$, the event-by-event fluctuation of the number of $b_1 b_2$ pairs. The grand canonical results correspond to small fluctuations, i.e., $\langle \delta N^2 \rangle / \langle N \rangle^2 \ll 1$, while the canonical description is necessary in the opposite limit.

IV. Master equation.—In this section, we formulate the general evolution equation which is valid for an arbitrary value of $\langle N \rangle$. It is a master equation for $P_n(\tau)$, the probability of finding n pairs of $b_1 b_2$ at time τ. This probability increases with time due to transitions from $n - 1$ and $n + 1$ states to the n state, while it also decreases due to transitions from the n state to $n - 1$ and $n + 1$ states. The transition probability $n \rightarrow n + 1$ per unit time due to pair creation is $G\langle N_{a_1} \rangle \langle N_{a_2} \rangle / V$ and the transition probability $n \rightarrow n - 1$ due to pair annihilation is $n^2 L/V$. Therefore, the master equation set has the form

$$\frac{dP_n}{d\tau} = \frac{G}{V} \langle N_{a_1} \rangle \langle N_{a_2} \rangle [P_{n-1} - P_n]$$

$$- \frac{L}{V} [n^2 P_n - (n + 1)^2 P_{n+1}], \qquad (21)$$

where $n = 0, 1, 2, 3, \ldots$. Multiplying the above equation by n and summing over n, one recovers Eq. (5), the general rate equation for the time evolution of the average number of $b_1 b_2$ pairs. However, the master equation (21) contains much more information than the rate equation (5). It contains enough information to solve for the evolution

of $\langle N \rangle$ (and all moments of N) for arbitrary $\langle N \rangle$. For example, one can also obtain the time evolution of particle fluctuations, $\langle \delta N^2 \rangle$, which are of physical importance for rarely produced particles.

We can convert the iterative equations (21) for P_n's into a partial differential equation for the generating function

$$g(x, \tau) = \sum_{n=0}^{\infty} x^n P_n(\tau). \qquad (22)$$

Multiplying Eq. (21) by x^n and summing over n, we find

$$\frac{\partial g(x, \tau)}{\partial \tau} = \frac{L}{V}(1 - x)(xg'' + g' - \epsilon g), \qquad (23)$$

where $g' \equiv \partial g / \partial x$. Note that the $g(1, \tau)$ does not change with time, which is equivalent to the conservation of total probability evident in Eq. (21).

The equilibrium solution, $g_{eq}(x)$, thus obeys the following equation:

$$xg_{eq}'' + g_{eq}' - \epsilon g_{eq} = 0. \qquad (24)$$

By variable substitution [$x = y^2/(4\epsilon)$] this equation can be reduced to the Bessel equation. The solution that is regular at $x = 0$ [since $g(0) = P_0 \leq 1$] is given by

$$g_{eq}(x) = \frac{1}{I_0(2\sqrt{\epsilon})} I_0(2\sqrt{\epsilon x}), \qquad (25)$$

where the normalization is fixed by $g(1) = \sum P_n = 1$.

The equilibrium probability distribution P_n can now be found from Eqs. (22) and (25) as

$$P_{n,eq} = \frac{\epsilon^n}{I_0(2\sqrt{\epsilon})(n!)^2}, \qquad (26)$$

and the value of the average number of $b_1 b_2$ pairs per event at equilibrium is given by

$$\langle N \rangle_{eq} = g_{eq}'(1) = \sqrt{\epsilon} \frac{I_1(2\sqrt{\epsilon})}{I_0(2\sqrt{\epsilon})}, \qquad (27)$$

which obviously coincides with the expected result for particle multiplicity in the canonical ensemble given by Eq. (18) and also reduces to the results for the two limiting cases in Sec. III. One can similarly formulate the master rate equation for particle distributions without the constraint by U(1) charge conservation as in a grand canonical system. In this case, the *quadratic* dependence on n in the second term of Eq. (21) will be replaced by a *linear* dependence and the equilibrium solution will be a Poisson distribution.

V. Conclusions.—We have formulated the kinetic master equation for strongly correlated production of particles, where the correlation is due to local charge conservation required by a U(1) internal symmetry. Our general rate equation is valid for an arbitrary value of $\langle N \rangle$; thus it reduces to the grand canonical results for large $\langle N \rangle$ and to the canonical results for small $\langle N \rangle$. Our equation provides a generalization of the standard rate equation beyond the grand canonical limit. We have shown that for rare particle

production the equilibrium abundance is much smaller and the relaxation time is much shorter than expected from the standard rate equation. For abundant particle production, where the standard rate equation applies to first order, one can use the general rate equations to study finite-size corrections to the grand canonical results.

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