Bosons in Cigar-Shaped Traps: Thomas-Fermi Regime, Tonks-Girardeau Regime, and In Between

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We present a quantitative analysis of the experimental accessibility of the Tonks-Girardeau gas in present-day experiments with cigar-trapped alkalis. For this purpose we derive, using a Bethe ansatz generated local equation of state, a set of hydrostatic equations describing one-dimensional, δ-interacting Bose gases trapped in a harmonic potential. The resulting solutions cover the *entire range* of atomic densities.

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Physics of trapped atomic gases attracted an enormous attention from both experimentalists and theorists in the past decade [1,2]. All atomic systems experimentally accessed so far can be well understood within a so-called mean-field picture, where the atom-atom correlations are weak and a particular atom can be effectively viewed as moving in a mean-field created by others and unaffected by the presence of the probe atom. Small deviations from the mean-field predictions can be calculated using the Bogoliubov approximation [3]. An interesting question arises as to what happens beyond the scope of the mean-field behavior. For the three-dimensional systems the far-beyond-mean-field regime has neither been reached experimentally (the JILA experimental group has made significant progress towards this goal [4]) nor understood theoretically.

For quasi-one-dimensional systems (cigar-shaped traps) the situation is different. Recall that a one-dimensional system of zero-range interacting bosons in a flat-bottom box is exactly integrable [5] via Bethe ansatz, for *all* values of the coupling strength. The exactly known equation of state in a box provides input for a unified beyond-mean-field treatment of trapped quasi-one-dimensional atomic gases, and that is what our paper is devoted to. The main goal of this paper is to provide quantitative criteria for an experimental realization of the Tonks-Girardeau gas [6] in atomic experiments, with the main emphasis on deviations of the spatial distribution from the mean-field Thomas-Fermi profile [7]. We paid special attention to the intermediate regime between weak and strong interactions, as the most realistic from an experimental point of view. The extreme case of infinitely strong interacting atoms in a harmonic potential was already investigated in Ref. [8]. Also the beyond-mean-field effects in the self-correlation function of the trapped one-dimensional gas were considered in [9].

Experimental progress in creating one-dimensional atomic gases is also quite fast. One-dimensional guiding of thermal cold atoms through elongated magnetic configurations [10] has already been successfully demonstrated. Furthermore, a recent experiment [11] has demonstrated truly one-dimensional behavior of a Bose-Einstein condensate trapped by a shallow magnetic gradient in the longitudinal direction combined with a blue detuned hollow laser beam, ensuring tight confinement in the radial directions. Another promising candidate in this respect is atoms in an array of 1D traps formed in the intersection of two far-detuned Gaussian standing waves [12], obtained in turn by releasing a full, three-standing-wave optical lattice, where up to $\sim$20% ground vibrational mode occupation is already experimentally accessible [13]. A 2D version of such an array (one standing wave) has already been reported in literature [14].

In highly elongated cigar-shaped traps ($\omega_\perp \gg \omega_z$), the transverse atomic motion is governed by the Hamiltonian

$$\hat{H}_\perp = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega_\perp^2(x^2 + y^2)}{2}, \qquad (1)$$

where m is the mass of the atoms, and $\omega_\perp$ and ω_z are the transverse and longitudinal frequencies of the trap, respectively. If neither temperature nor interaction energy per particle ϵ (defined below) exceeds the transverse level spacing $\hbar\omega_\perp$, atoms occupy the ground mode of this Hamiltonian, and the system becomes effectively one dimensional [15]. In particular, the interparticle interaction in the longitudinal direction can be well approximated by an effective two-atom interaction potential [15]

$$U_{1D}(z) = g_{1D}\delta(z). \qquad (2)$$

where $g_{1D} = -\hbar^2/\tilde{\mu}a_{1D}$ is an effective one-dimensional coupling constant, and $a_{1D} = (-a_\perp^2/2a)[1 - C(a/a_\perp)]$ is the one-dimensional scattering length defined analogously to the three-dimensional case as $a_{1D} = -\partial\Delta/\partial k_z|_{k_z\to 0^+}$, with $\Delta(k_z)$ being the scattering phase of the even scattered wave. Here and below, $\tilde{\mu} = m/2$ is the reduced mass, $a_\perp = \sqrt{\hbar/\tilde{\mu}\omega_\perp}$ is the size of the ground state of the transverse Hamiltonian in Eq. (1), a is the three-dimensional scattering length, and $C = \lim_{s\to\infty}(\int_0^s ds'/\sqrt{s'} - \sum_{s'=1}^{s} 1/\sqrt{s'}) = 1.4603\ldots$.

Up to terms containing C, the effective one-dimensional interaction is a simple projection of the

three-dimensional zero-range interaction on the transverse ground mode. Transverse renormalization effects captured in the C-dependent terms become important only for strong confinement, $|a| \gg a_\perp$.

In addition to transverse trapping and interparticle interactions, the atoms are also subject to weak residual longitudinal trapping, which is usually represented by a one-dimensional harmonic potential of a frequency ω_z. The effective one-dimensional Hamiltonian for N trapped atoms is thus

$$\hat{H}_{1D} = \hat{H}^0_{1D} + \sum_{j=i}^{N} \frac{m\omega_z^2 z_i^2}{2}, \tag{3}$$

where

$$\hat{H}^0_{1D} = -\frac{\hbar^2}{2m} \sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g_{1D} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \delta(z_i - z_j) \tag{4}$$

is the well-known Hamiltonian for a one-dimensional, δ-interacting Bose gas in a flat-bottom box. This Hamiltonian can be diagonalized via Bethe ansatz [5], and thus the equation of state of such a gas is known *exactly* for all densities and temperatures [16]. At zero temperature the energy per particle $\epsilon(n)$ is given through

$$\epsilon(n) = \frac{\hbar^2}{2m} n^2 e(\gamma(n)), \tag{5}$$

where the dimensionless parameter $\gamma = 2/n|a_{1D}|$ is inversely proportional to the one-dimensional gas parameter $n|a_{1D}|$, n is the one-dimensional number density of particles, and the function $e(\gamma)$ is given by

$$e(\gamma) = \frac{\gamma^3}{\lambda^3(\gamma)} \int_{-1}^{1} g(x \mid \gamma) x^2 \, dx. \tag{6}$$

Functions $g(x \mid \gamma)$ and $\lambda(\gamma)$ are solutions of the Lieb-Liniger system of equations [5]:

$$g(x \mid \gamma) - \frac{1}{2\pi} \int_{-1}^{1} \frac{2\lambda(\gamma)}{\lambda^2(\gamma) + (y - x)^2} g(y \mid \gamma) \, dy = \frac{1}{2\pi}, \tag{7}$$

$$\lambda(\gamma) = \gamma \int_{-1}^{1} g(x \mid \gamma) \, dx. \tag{8}$$

Limits of small and large gas parameter $n|a_{1D}|$ can be expressed in closed form (see [5]) as

$$n|a_{1D}| \to 0, \qquad \epsilon(n) \to \frac{\pi^2 \hbar^2}{6m} n^2, \tag{9}$$

and

$$n|a_{1D}| \to \infty, \qquad \epsilon(n) \to \frac{1}{2} g_{1D} n. \tag{10}$$

The low-density limit (9) corresponds to the case of infinitely strong interactions; the corresponding atomic system is usually referred to as a gas of impenetrable bosons or Tonks-Girardeau gas [6]. Notice that the expression for the energy per particle formally coincides with the one for free fermions: this is a manifestation of the Fermi-Bose duality in one-dimensional systems [17]. The opposite limit of high densities (10) represents (if multiplied by the number of particles and integrated over volume) the Thomas-Fermi energy functional first introduced by Bogoliubov [3]. It is usually viewed as the thermodynamic limit of the Gross-Pitaevskii energy functional [18], derived in turn on the basis of the mean-field approximation. It should be mentioned that in the three-dimensional case the mean-field approximation is valid for low densities, contrary to the one-dimensional case where it is valid at high densities instead.

We now introduce the classical hydrodynamical approximation: we suppose that the trapped gas at each position z is in local thermal equilibrium, with local energy per particle given through Eq. (5). At zero temperature the hydrodynamic equations of motion read

$$\begin{aligned} &\frac{\partial}{\partial t} n + \frac{\partial}{\partial z} (n v) = 0, \\ &\frac{\partial}{\partial t} v + v \frac{\partial}{\partial z} v = -(1/m) \frac{\partial}{\partial z} [\phi(n) + V(z)], \end{aligned} \tag{11}$$

where v is the atomic velocity, $V(z) = m\omega_z^2 z^2/2$ is the potential energy per particle, and

$$\phi(n) \overset{T=0}{=} \left(1 + n \frac{\partial}{\partial n}\right) \epsilon(n) \tag{12}$$

is the Gibbs free energy per particle obtained using Eqs. (5)–(8), (11), and (12) and tabulated in Ref. [19]. The validity of the hydrodynamical approximation requires that the typical energy per particle $\epsilon \sim mv^2/2 + \phi + V$ be much higher than the longitudinal quantum level spacing $\hbar\omega_z$.

In what follows we will be interested in a *steady-state* solution of the system (11). The first integral of the stationary version of this system obviously reads

$$\begin{aligned} \phi(n) + V(z) &= \mu \quad \text{for } |z| \le R, \\ n &= 0 \quad \text{for } |z| > R, \end{aligned} \tag{13}$$

where μ can be proven to be the chemical potential of the system, fixed by a normalization condition

$$\int_{-R}^{+R} n(z) \, dz = N. \tag{14}$$

Here the atomic cloud radius R is given by $V(R) = \mu$.

In the limit of low density (9) the density profile is a square root of parabola [8]

$$n(z) = n^0_{\text{Tonks}} \left(1 - \frac{z^2}{R^2_{\text{Tonks}}}\right)^{1/2} \tag{15}$$

for $z \in [-R_{\text{Tonks}}, R_{\text{Tonks}}]$, with $n = 0$ elsewhere, where

$$n^0_{\text{Tonks}} = [(2/\pi^2) N (m\omega_z/\hbar)]^{1/2}, \tag{16}$$

$$R_{\text{Tonks}} = [2N(\hbar/\omega_z m)]^{1/2}. \tag{17}$$

The opposite limit (10) reproduces the familiar Thomas-Fermi parabola [7]. In this limit, it is easy to show that

$$n(z) = n_{\mathrm{TF}}^0\left(1 - \frac{z^2}{R_{\mathrm{TF}}^2}\right) \tag{18}$$

for $z \in [-R_{\mathrm{TF}}, R_{\mathrm{TF}}]$, with $n = 0$ elsewhere, where

$$n_{\mathrm{TF}}^0 = [(9/64)N^2(m\omega_z/\hbar)^2|a_{1\mathrm{D}}|]^{1/3}, \tag{19}$$

$$R_{\mathrm{TF}} = [3N(\hbar/m\omega_z)^2/|a_{1\mathrm{D}}|]^{1/3}. \tag{20}$$

In general, the rescaled density $s = n/n_{\mathrm{TF}}^0$ satisfies the rescaled version of the system (13) and (14),

$$\frac{1}{2\eta} f\left(\frac{2}{\eta s(x)}\right) + \frac{1}{2}x^2 = \frac{b^2}{2}, \tag{21}$$

$$\int_{-b}^{b} s(x)\,dx = \frac{4}{3}, \tag{22}$$

where $x = z/R_{\mathrm{TF}}$ is the dimensionless coordinate, and $f(\gamma) = [3e(\gamma) - \gamma de/d\gamma]/\gamma^2$ is the dimensionless Gibbs energy (see [19]). We see that this system has only one governing parameter,

$$\eta \equiv n_{\mathrm{TF}}^0|a_{1\mathrm{D}}| = \frac{(9/2)^{1/3}}{8}\left(\frac{a_\perp^4 mN\omega_z(1 - Ca/a_\perp)^2}{a^2\hbar}\right)^{2/3}, \tag{23}$$

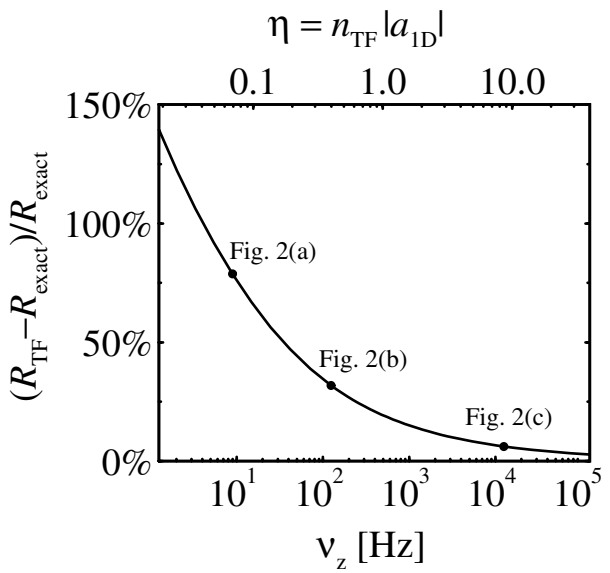

FIG. 1. Discrepancy between the cloud size computed from the exact equation of state and from the Thomas-Fermi prediction, as a function of the (single) governing parameter η (23) shown in the upper x axis. As an example of a "physical" variation of the parameter η we consider $N = 200$ confined in a cigar-shaped trap of transverse frequency $\nu_\perp = 180$ kHz ^{133}Cs atoms, whose scattering length is magnetically tuned [20] to $a = +120a_{\mathrm{Bohr}}$. Variation of the parameter η is supposed to be realized through the variation of the longitudinal frequency ν_z (lower x axis). For the three representative points marked in this figure, we computed the density profiles from the exact Thomas-Fermi and Tonks-Girardeau equations of state. Results are shown in Fig. 2.

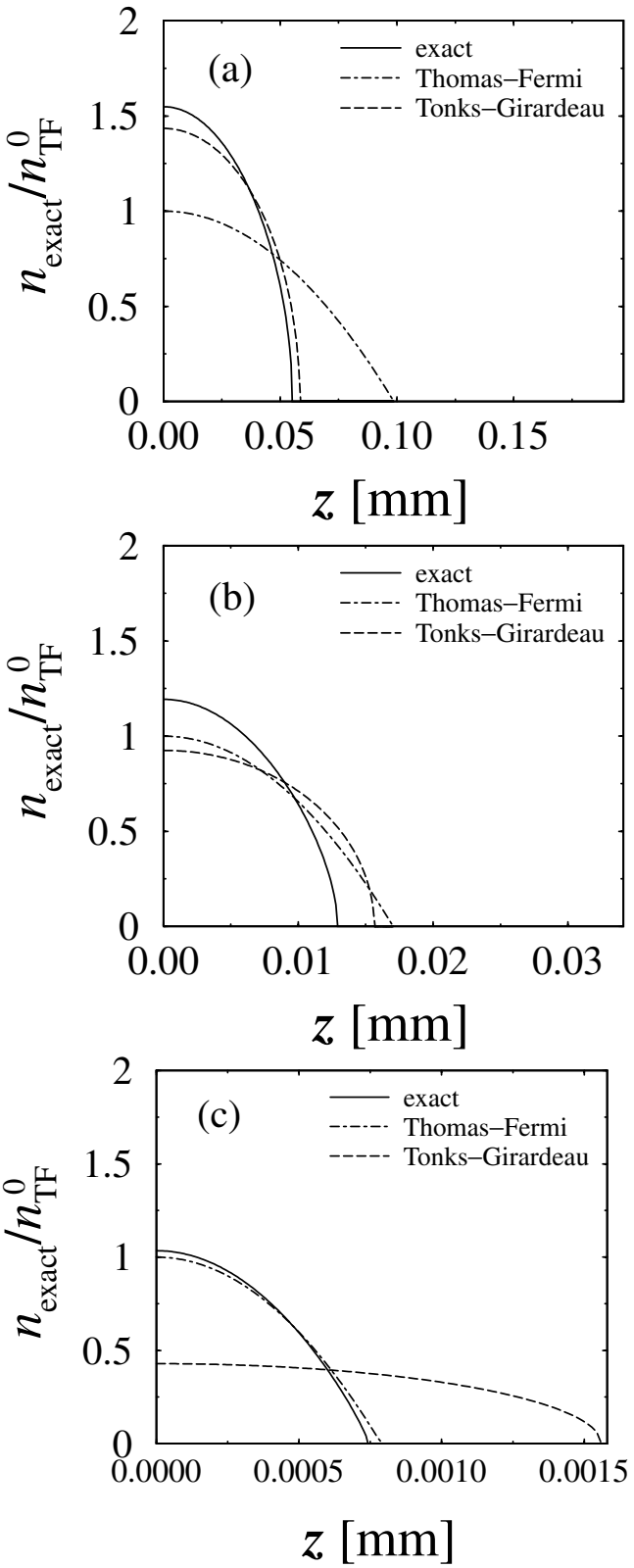

FIG. 2. Comparison of density profiles produced by Thomas-Fermi, exact, and Tonks-Girardeau local equations of state for the points indicated in Fig. 1. (a) $\eta = 0.07$, $\nu_z = 10$ Hz; Tonks-Girardeau regime. (b) $\eta = 0.4$, $\nu_z = 130$ Hz; intermediate regime. (c) $\eta = 9$, $\nu_z = 13$ kHz; Thomas-Fermi regime. The number of atoms, three-dimensional scattering length, and transverse frequency are the same as in Fig. 1.

whose physical meaning is the Thomas-Fermi estimate of the one-dimensional gas parameter in the center of the trap. It is easy to show that the limit of small η corresponds to the Tonks-Girardeau limit, and the limit of large η, to the Thomas-Fermi limit:

$$\eta \ll 1 \longrightarrow \text{Tonks-Girardeau profile} \tag{24a}$$

[see Eq. (15)],

$$\eta \gg 1 \longrightarrow \text{Thomas-Fermi profile} \tag{24b}$$

[see Eq. (18)].

The exactly known equation of state [Eq. (5)] allows one to precisely map the transition between these two limiting behaviors. A convenient way to detect, both experimentally and numerically, the degree to which the system is in one or the other limiting regime is to compare the size of the cloud of trapped atoms [R computed from Eqs. (13) and (14)] to the Thomas-Fermi prediction R_{TF} in Eq. (20). This is shown in Fig. 1. Density profiles for three representative values of the governing parameter η are plotted in Fig. 2. As Figs. 1 and 2 make clear, the transition between the two limiting regimes happens over the range of the governing parameter η from about 0.1 to 10, reaching 75% of the cloud size discrepancy for the former.

Let us summarize the requirements for a successful observation of the zero-temperature Tonks-Girardeau density profile (15): (i) The coupling constant $g_{1\mathrm{D}}$ [see (2)] must be positive. This condition is equivalent to $0 < a < a_\perp/C$. (ii) To ensure the one-dimensional behavior of the system the energy per particle must not exceed the transverse level spacing; in the case of the Tonks-Girardeau gas this condition leads to $\hbar\omega_z N \ll \hbar\omega_\perp$. (iii) The number of particles must be well below the Tonks-Girardeau vs Thomas-Fermi boundary which requires $\eta \ll 1$, where η is given by (23). (iv) It turns out that not every set of trap and atom parameters for the energy per particle at the Tonks-Girardeau vs Thomas-Fermi boarder ($\eta \sim 1$) is consistent with the hydrodynamical approximation, i.e., exceeds the longitudinal level spacing. To avoid that, one must require $\hbar^2/m|a_{1\mathrm{D}}|^2 \gg \hbar\omega_z$ or $a \gg (a_\perp^2/a_z)(1 - Ca/a_\perp)$, where $a_z \sim (\hbar/m\omega_\perp)^{1/2}$ is the size of the longitudinal ground state. If the above condition is violated, then, as the number of particles decreases, the system will pass from the Thomas-Fermi regime to the ideal gas directly, skipping the Tonks-Girardeau regime. (v) One may also require that, when one is already in the Tonks-Girardeau regime ($\eta \ll 1$), the number of atoms must nevertheless be high enough to ensure the hydrodynamical, nonideal gas behavior. It turns out that this requirement leads to a trivial condition of having more than one atom in the system: $N \gg 1$.

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