Effect of Trapped Ions on Shielding of a Charged Spherical Object in a Plasma

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Collisions have traditionally been neglected in calculating the shielding around a small spherical collector in a plasma, and the plasma flow to the collector. We show analytically that, in dusty plasmas under typical discharge conditions, ion charge-exchange collisions lead to the buildup of negative-energy trapped ions which dominate the shielding cloud in the nonlinear region near a dust grain and substantially increase the ion current to the grain, even when the mean-free path is much greater than the Debye length.

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Introduction.—The problem of electrostatic shielding around a small spherical collector immersed in a nonstreaming plasma, and the related problem of electron and ion flow to the collector, date to the very origins of plasma physics. The initial work by Langmuir and collaborators [1] in the 1920s was directed toward understanding of electrostatic probes. This was followed by important papers in every subsequent decade [2–9], in some cases aimed at understanding sheaths around spacecraft, and more recently by the physics of dusty plasmas [5–15]. "Dust grains," i.e., micron-scale particles, typically acquire a negative charge of the order of thousands of electron charges when immersed in a plasma. As a result, they interact strongly with each other, exhibiting a variety of interesting collective behaviors. All studies of dusty plasma begin with a model for the grain charge, which in steady state is determined by the requirement that the electron current to the grain be balanced by an equal and opposite ion current (the floating condition), and a model for the shielded interactions of grains with each other and with external forces.

For over seventy years, theoretical calculations of shielding and grain charging have neglected electron and ion collisions. The initial justification [1] for this was that the mean-free path $\lambda_{\rm mfp}$ in a discharge is typically long compared to other lengths of interest. When collisions are neglected, all of the ions and electrons in the vicinity of the collector, in steady state, must be positive energy particles which come in from the ambient plasma ($r = \infty$) and either hit the collector (in which case it is usually assumed that they are absorbed) or fly back out to $r = \infty$. However, Bernstein and Rabinowitz [3] pointed out in 1959 that negative-energy positive-charge ions can be created if there are occasional collisions in which ions lose energy. These ions will be trapped in the potential well around a negatively charged collector, and, even if the collision frequency ν is very small, the trapped ion density $n_t(r)$ will build up indefinitely until it is limited by other collisional processes which result in the loss of trapped ions [3,4,6,9]. Goree [9] noted the density of trapped ions is *independent* of ν in steady state, since the creation and loss rates are both proportional to ν. Many authors commented on the probable importance of trapped ion effects, but analytic theories continued to be collisionless, because it was generally [3,4,6,9,10] thought that inclusion of trapped ions would render the problem intractable. However, in 1992, Goree confirmed in a Monte Carlo simulation that the total number of trapped ions can be quite significant, and very recently Zobnin *et al.* [10] used Monte Carlo simulations to calculate the trapped ion density profile $n_t(r)$ and the self-consistent potential $\phi(r)$ for the first time.

In this paper, we introduce a fully analytic method for calculating the distribution of trapped as well as untrapped ions, and we solve self-consistently for $n_t(r)$, $\phi(r)$, and the untrapped ion density $n_u(r)$. We show that under typical conditions the inner part of the shielding cloud is made up primarily of trapped ions, and that $\phi(r)$ is thus different from the results of the collisionless theories. The presence of trapped ions also significantly increases the ion flow to the grain. Our analytic results appear to be in general agreement with the Monte Carlo calculations of Zobnin *et al.* [10], and provide a general theoretical framework for elucidating the plasma response to a charged grain.

It should be noted that in this paper we consider only the case of an isotropic, nonflowing plasma, which is relevant to dust grains in bulk plasma, e.g., in microgravity experiments. In laboratory discharges, dust typically resides in or near the sheath, where there are strong ion flows that break the symmetry assumed here. We are looking into the extension of the present methods to that case.

Model and calculation.—We consider a single stationary grain of radius a, immersed in a nonflowing plasma consisting of singly charged positive ions and neutral molecules, each assumed Maxwellian with temperature T, and Maxwellian electrons with temperature T_e. The ambient plasma density is n_0. We assume the following: (i) steady state; (ii) $a \ll \lambda_D$, where $\lambda_D \equiv [4\pi n_0 e^2 (T_e^{-1} + T^{-1})]^{-1/2}$ is the Debye length; (iii) ions are subject only to charge-exchange collisions with neutrals, with an energy-independent collision frequency ν; (iv) ν is small, in the sense that the probability of a collision is small during the time for an untrapped ion to traverse the potential well, or for a trapped ion to make one rotation in its orbit. Roughly speaking, this is equivalent to the assumption that $\lambda_{\rm mfp} \gg \lambda_D$; (v) absence of significant centrifugal potential barriers, the primary assumption of orbital-motion-limited (OML) theory [3–8,11–12], discussed below in more detail.

Trapped ions are created by ion-neutral charge-exchange collisions. Every time a collision occurs, the old ion disappears, and a new ion is created whose velocity is chosen at random from the neutral molecule distribution function $\exp(-mv^2/2T)$. Consider the class of trapped ions which were created by collisions which occurred at radial location r', and let $h(r, v, \theta; r')$ be the phase-space distribution function of these ions. Here, r is the present location and $\mathbf{v} \equiv (v, \theta)$ is the present velocity of the ion; θ is the angle between $\mathbf{r}$ and $\mathbf{v}$. Because of spherical symmetry, $h(r, v, \theta; r')$ does not depend on the angular coordinates of $\mathbf{r}$ and $\mathbf{r}'$, nor on the azimuthal coordinate of $\mathbf{v}$. If trapped ions were collisionless, then the steady-state Vlasov equation would tell us that the distribution function $h(r, v, \theta; r')$, for a given birthplace r', is a function only of the constants of the motion, total energy $\varepsilon \equiv \frac{1}{2}mv^2 + e\phi(r)$ and angular momentum $L = mvr\sin\theta$. Since the ions are born Maxwellian, this distribution must be of the form

$$h(r, v, \theta; r') = C(r') \exp\left(-\frac{mv^2}{2T} - \frac{e\phi(r)}{T}\right). \qquad (1)$$

Actually, trapped ions do undergo charge-exchange collisions, and every time this happens an ion is lost from $h(r, v, \theta; r')$. But we have assumed that the collision frequency ν is energy independent, and that the time between collisions ν^{-1} is long compared to the orbit period of a trapped ion. Thus the correlation between r and r' is lost, and any ion in $h(r, v, \theta; r')$ is equally likely to have been lost to a collision. So $h(r, v, \theta; r')$ must be of the form (1), even with collisions.

However, the Maxwellian distribution (1) is not populated for every value of v and θ. Several conditions must be satisfied. First, the ion must have negative total energy, i.e.,

$$\tfrac{1}{2}mv^2 < -e\phi(r) \equiv \tfrac{1}{2}mv_1^2(r), \qquad (2a)$$

otherwise it is not a trapped ion. A second condition is that the total energy ε must be greater than $e\phi(r')$, since the ion was born at r' with positive kinetic energy, i.e.,

$$e\phi(r') - e\phi(r) \le \tfrac{1}{2}mv^2. \qquad (2b)$$

A third condition is that the ion must have enough angular momentum that its trajectory does not intercept the grain radius a. (Since we assume that the trapped ion period is short compared to the collision time, we treat ions which intercept the grain as if they are lost immediately, and just delete them from the trapped ion distribution.) This leads to a requirement on v,

$$[e\phi(r) - e\phi(a)]\frac{a^2}{r^2 - a^2} \le \tfrac{1}{2}mv^2, \qquad (2c)$$

and a requirement on θ,

$$\sin\theta > \frac{a}{r}\sqrt{1 + \frac{2[e\phi(r) - e\phi(a)]}{mv^2}} \equiv \sin\theta_0(r, v). \qquad (2d)$$

We can combine (2b) and (2c) into a condition $v_0^2(r, r') \le v^2$, where $v_0^2(r, r')$ is the larger of the left-hand sides. Finally, the phase space coordinates (r, v, θ) must be accessible from r', i.e., there must not be any barrier in the radial *effective potential* $U(r) \equiv e\phi(r) + L^2/mr^2$ which prevents an ion born at r', with angular momentum $L = mvr\sin\theta$ and energy $\varepsilon \equiv \frac{1}{2}mv^2 + e\phi(r)$, from reaching r. We shall assume that there are no significant potential barriers. This is the standard assumption used in OML theory [3–8,11–12]. We have recently shown [12,13], that it is well satisfied for untrapped ions, for $a \ll \lambda_D$, and other conditions typical of dusty plasma. Using the same approach as in Refs. [12,13], we can show that the assumption is also valid for trapped ions; details will be presented in a subsequent publication.

The total number of trapped ions which were born at r', which we shall call $4\pi r'^2 g(r')$, is thus given by integrating $h(r, v, \theta; r')$ over (r, v, θ) with the conditions (2). Every time one of these ions undergoes a collision, it is lost from $h(r, v, \theta; r')$; thus, the loss rate is $4\pi r'^2 g(r')\nu$. In steady state, we can set this loss rate equal to the rate at which trapped ions are created by collisions (of either trapped ions or untrapped ions) at r'. This condition determines the factor $C(r')$ in Eq. (1), giving

$$h(r, v, \theta; r') = \frac{1}{4\pi^{5/2}}\left(\frac{2T}{m}\right)^{-3/2}[n_u(r') + n_t(r')] \times e^{-(mv^2)/(2T) + [e\phi(r') - e\phi(r)]/T} \times \frac{G(r', r')}{\int_a^\infty dr''\, r''^2 G(r'', r')}, \qquad (3)$$

provided $\sin\theta > \sin\theta_0(r, v)$ and $v_0(r, r') < v < v_1(r)$. Here,

$$G(r, r') \equiv e^{-[e\phi(r)]/T}\int_{v_0(r, r')}^{v_i(r)} dv\, v^2 e^{-(mv^2)/(2T)} \times \cos[\theta_0(r, v)]. \qquad (4)$$

Now $n_u(r)$ is obtained from the result of OML theory [15],

$$n_u(r) = n_0\frac{2}{\sqrt{\pi}}\exp\left(-\frac{e\phi(r)}{T}\right)\int_{\sqrt{-e\phi(r)/T}}^\infty dt\, t^2 e^{-t^2} \times \left[1 + \sqrt{1 - \frac{a^2}{r^2}\left(1 + \frac{e[\phi(r) - \phi(a)]}{Tt^2}\right)}\right], \qquad (5)$$

where the integral is taken only over values of t such that the argument of the square root is positive. The electron density $n_e(r)$ is well approximated [4] by a Boltzmann factor $n_e(r) = n_0 \exp[e\phi(r)/T_e]$. The trapped ion density $n_t(r)$ is obtained by integrating $h(r, v, \theta; r')$ over $(v, \theta; r')$. Since $n_t(r')$ also appears as a source term on the right-hand side (rhs) of Eq. (3), this procedure actually yields a linear integral equation for $n_t(r)$,

$$n_t(r) = \int_a^\infty dr' K(r, r') n_t(r') + \int_a^\infty dr' K(r, r') n_u(r'), \tag{6}$$

where

$$K(r, r') = \frac{4}{\pi^{1/2}} \left(\frac{2T}{m}\right)^{-3/2} \frac{r'^2 e^{[e\phi(r')]/T} G(r', r') G(r, r')}{\int_a^\infty dr'' r''^2 G(r'', r')}. \tag{7}$$

To complete the calculation, it is necessary to solve Poisson's equation,

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi}{dr} = 4\pi e[n_u(r) + n_t(r) - n_e(r)], \tag{8}$$

self-consistently with Eqs. (5)–(7). If the collector is a probe biased to a specified potential ϕ_0, the boundary conditions for (8) are $\phi(\infty) = 0$, and $\phi(a) = \phi_0$. If the collector is a dust grain, $\phi(a)$ is set equal to the floating potential ϕ_f, i.e., the value for which there is no net electric current to the grain. In the limit of small ν, ϕ_f is determined [14] by

$$\left(1 - \frac{e\phi_f}{T_i}\right) \exp\left(-\frac{e\phi_f}{T_e}\right) = \left(\frac{m_i T_e}{m_e T}\right)^{1/2}, \tag{9}$$

typically $\phi_f \sim -3T_e$. For finite values of ν, there are corrections to the ion flow to the grain, and therefore to ϕ_f, as will be discussed below.

A procedure for doing this calculation is as follows. Begin with the known solution [15] from OML theory [Eqs. (5) and (8)] for $\phi(r)$ and $n_u(r)$ if there are no trapped ions. Calculate a first approximation $n_t^{(1)}(r)$ to $n_t(r)$ by neglecting the first term on the rhs of Eq. (6). This can be interpreted as the population of "first generation" trapped ions created by collisions of untrapped ions. Recalculate $\phi(r)$ and $n_u(r)$ from Eqs. (8) and (5). Then calculate a second iterate $n_t^{(2)}(r)$ by using $n_t^{(1)}(r)$ in the first term on the rhs of (6). $n_t^{(2)}(r)$ can be regarded as the population of trapped ions created by either the collision of an untrapped ion or of a first-generation trapped ion. Proceed with this iteration scheme to convergence.

Results and discussion.—We present the solution of Eqs. (5)–(9) for a case where the collector is a dust grain at floating potential, $T/T_e = 0.04$, and $a/\lambda_D = 0.015$. Figure 1 shows the density $n_t(r)$ of trapped ions (solid curve), and the deviations of the untrapped ion density (dashed curve) and electron density (dotted curve) from the ambient value, $\Delta n_u(r) \equiv n_u(r) - n_0$ and $\Delta n_e(r) \equiv n_e(r) - n_0$. Notice that $n_t \gg \Delta n_u \gg \Delta n_e$ near the grain. For this case, trapped ions dominate the shielding around the charged grain out to $r = 0.7\lambda_D$. In Fig. 2, we show $Q_t(r)$, the trapped ion charge enclosed within radius r (solid curve), and $Q_u(r)$, $Q_e(r)$, respectively, the deviation of the untrapped ion charge (dashed curve) and of the electron charge (dotted curve) within radius r from the ambient value. All of the Q's are scaled to the charge on the grain. Note that $Q_t > Q_u$ out to $r \approx 1.4\lambda_D$, where the grain charge is 42% neutralized. In Fig. 3, we plot $r\phi(r)$. On this semilog plot, an unshielded Coulomb potential would appear as a horizontal straight line, and the Debye-shielded potential would appear as the oblique dotted line. In the absence of trapped ions (dashed curve), $\phi(r)$ differs noticeably from the Debye-shielded potential for all r. Trapped ions add just enough additional shielding to bring the potential to very nearly the Debye shielded

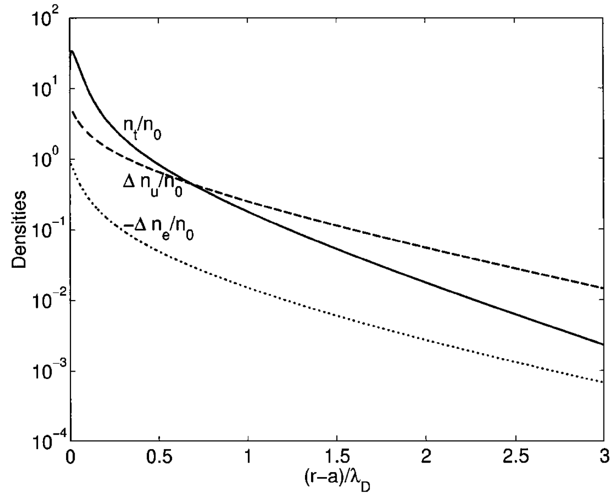

FIG. 1. Trapped ion density [$n_t(r)$, solid line], deviation of untrapped ion density from ambient [$\Delta n_u(r)$, dashed line], and deviation of electron density from ambient [$\Delta n_e(r)$, dotted line], all scaled to ambient density n_0.

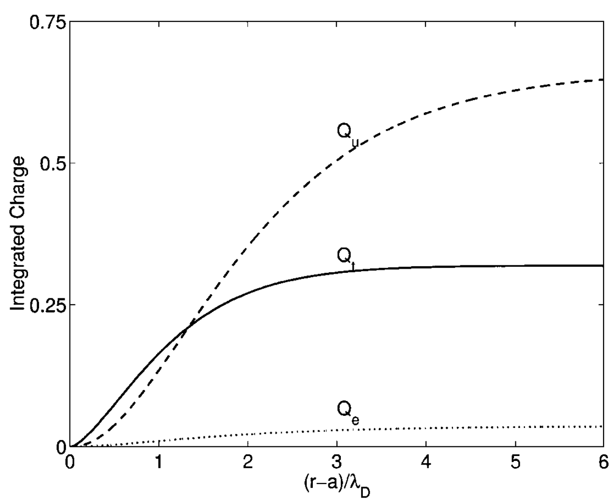

FIG. 2. Integrated trapped ion charge from $r = a$ to r [$Q_t(r)$, solid line], deviation of the integrated untrapped ion charge from ambient [$Q_u(r)$, dashed line], and deviation of the integrated electron charge from ambient [$Q_e(r)$, dashed line], all scaled to the charge on the grain.

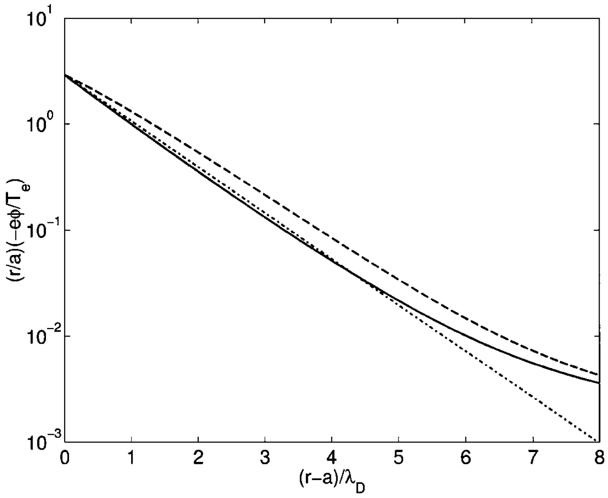

FIG. 3. Plots of $-(r/a)e\phi(r)/T_e$ for three different models: self-consistent potential including trapped ions (solid line), potential with trapped ions neglected (dashed line), and Debye potential (dotted line).

form for $0.5\lambda_D < r < 5\lambda_D$ [At very large r, $\phi(r)$ must fall off as r^{-2}.] [3,8].

The theory presented above depends on the dimensionless parameters T/T_e and a/λ_D. In the typical situation for dusty plasmas in discharges, where $a^2/\lambda_D^2 \ll T/T_e \ll 1$, it is clear that in steady state the trapped ion population must exceed the untrapped ion population in the region where there is a strong potential well, i.e., $-e\phi(r) > T_i$, since nearly every collision of an untrapped ion results in the creation of a trapped ion, but only a small fraction of the collisions of trapped ions result in the loss of a trapped ion. If $T/T_e < a^2/\lambda_D^2$, the trapped ion population falls off, because many of the newly born ions will have low angular momentum and immediately fall onto the grain. In the opposite limit, where T/T_e approaches unity, the trapped ion population again falls off, because many of the newly born ions will have enough energy to escape to $r = \infty$. These parametric dependences will be discussed more extensively in future literature.

We note that $n_t(r)$ and $\phi(r)$ do not depend on the value of ν, in the limit of small ν. However, collisions also modify the ion current F_i to the grain, and this effect is proportional to ν. An untrapped ion coming in from $r = \infty$, whose trajectory would have taken it into the grain, may experience a collision before it reaches the grain. This represents a reduction $-\Delta F_{i1}$ in the collisionless ion flux F_{i0} to the grain. On the other hand, after a collision (involving either an untrapped ion or a trapped ion), the newly born ion may have low angular momentum and fall onto the grain. This represents an increase ΔF_{i2} in the ion flow to the grain. Both ΔF_{i1} and ΔF_{i2} are proportional to ν, and thus might be thought to be small. However, it is clear that, under typical conditions $\Delta F_{i1} \ll \Delta F_{i2}$, since ΔF_{i2} represents very nearly the entire loss rate of trapped ions (trapped ions are usually lost via collisions that cause the new ion to fall onto the grain, only occasionally by collisions that boost the new ion into the untrapped population), whereas ΔF_{i1} represents only a small fraction of the creation rate of trapped ions (those that result from the collision of an untrapped ion which would have hit the grain). Thus, collisions result in an increase in ion flow to the grain, and this increase is in fact large, even for quite small values of ν, since it is multiplied by a large factor of the order of n_t. This change in F_i can substantially suppress the floating potential ϕ_f. Calculations of F_i and ϕ_f have been performed and will be presented in future literature.

In conclusion, we wish to point out that the presence of a large population of trapped ions can profoundly change the interaction of a grain with other grains, and with external forces. We have previously argued [15] that shielding by untrapped ions cannot lead to a net attractive electrostatic force between negatively charged grains. But a grain with its trapped ion cloud can behave similar to a "classical atom"; the trapped ion cloud can be polarized, thereby shielding the grain from electric fields [6,9], and possibly leading to van-der-Waals–type attractive forces between grains. We are in the process of calculating these effects.

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