Exact Universal Amplitude Ratios for Two-Dimensional Ising Models and a Quantum Spin Chain

N. Sh. Izmailian[1,2] and Chin-Kun Hu[1,*]
[1]Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
[2]Yerevan Physics Institute, Yerevan 375036, Armenia
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Let f_N and ξ_N^{-1} represent, respectively, the free energy per spin and the inverse spin-spin correlation length of the critical Ising model on a $N \times \infty$ lattice, with $f_N \to f_\infty$ as $N \to \infty$. We obtain analytic expressions for a_k and b_k in the expansions $N(f_N - f_\infty) = \sum_{k=1}^{\infty} a_k/N^{2k-1}$ and $\xi_N^{-1} = \sum_{k=1}^{\infty} b_k/N^{2k-1}$ for square, honeycomb, and plane-triangular lattices, and find that $b_k/a_k = (2^{2k} - 1)/(2^{2k-1} - 1)$ for all of these lattices, i.e., the amplitude ratio b_k/a_k is universal. We also obtain similar results for a critical quantum spin chain and find that such results could be understood from a perturbated conformal field theory.

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Experimental data, analytical and simulational studies of phase transition models, and renormalization group (RG) theory suggest that critical systems can be grouped into universality classes so that the systems in the same class have the same set of critical exponents [1–3]. RG theory was also used to propose that critical systems of the same universality class could have universal finite-size scaling functions (UFSSF's) and universal amplitude ratios [2–4], and some analytical and numerical calculations of critical systems have supported the idea of universal amplitude ratios [2–4]. By using Monte Carlo methods [5] and choosing appropriate aspect ratios for lattices of critical systems, Hu *et al.* found UFSSF's for percolation and Ising models [6]. Despite the success of RG theory and Monte Carlo simulations, it is valuable to have more analytical results which could widen or deepen our understanding of the universality of critical systems. In this Letter we present exact calculations for a set of universal amplitude ratios for the Ising model on square (sq), honeycomb (hc), and plane-triangular (pt) lattices [7–9] and for a quantum spin chain [10], which is in the universality class of two-dimensional (2D) Ising models. As far as we know, no previous RG arguments, analytical calculations, or numerical studies predict the existence of this whole set of universal amplitude ratios.

Let f_N and ξ_N^{-1} represent, respectively, the free energy per spin and the inverse spin-spin correlation length of the Ising model [7–9] on an $N \times \infty$ lattice with periodic boundary conditions, with $f_N \to f_\infty$ as $N \to \infty$. In this Letter, we obtain analytic equations for a_k and b_k in the expansions,

$$N(f_N - f_\infty) = \sum_{k=1}^{\infty} \frac{a_k}{N^{2k-1}}, \tag{1}$$

$$\xi_N^{-1} = \sum_{k=1}^{\infty} \frac{b_k}{N^{2k-1}}, \tag{2}$$

for sq, hc, and pt lattices, and find that

$$b_k/a_k = (2^{2k} - 1)/(2^{2k-1} - 1), \tag{3}$$

for all of these lattices, i.e., the amplitude ratio a_k/b_k is universal. We also obtain similar expansions for the critical ground state energy E_0 and the critical first energy gap $(E_1 - E_0)$ of a quantum spin chain [10], which are, respectively, the quantum analogies of the free energy and inverse spin-spin correlation length for the Ising model, and find that the amplitude ratios have the same values. We could physically understand such results from a perturbated conformal field theory.

Consider an Ising ferromagnet on an $N \times M$ lattice with periodic boundary conditions (i.e., a torus). The Hamiltonian of the system is

$$\beta H = -J \sum_{\langle ij \rangle} s_i s_j, \tag{4}$$

where $\beta = (k_B T)^{-1}$, the Ising spins $s_i = \pm 1$ are located at the sites of the lattice, and the summation goes over all nearest-neighbor pairs of the lattice. We consider a transfer matrix acting along the M direction [11–13]. If Λ_0 and Λ_1 are the largest and the second-largest eigenvalues of the transfer matrix, in the limit $M \to \infty$ the free energy per spin, f_N, and the inverse longitudinal spin-spin correlation length, ξ_N^{-1}, are

$$f_N = \frac{1}{\zeta N} \ln \Lambda_0 \quad \text{and} \quad \xi_N^{-1} = \frac{1}{\zeta} \ln(\Lambda_0/\Lambda_1). \tag{5}$$

Here ζ is a geometric factor which is 1, $2/\sqrt{3}$, and $1/\sqrt{3}$ for sq, hc, and pt lattices, respectively [2]. Exact expressions for eigenvalues Λ_0 and Λ_1 are available for all lattices under consideration: sq [7,11–13], hc [2,9], and pt [8].

We start from the Ising model on the sq lattice. Onsager [7] has obtained expressions for all eigenvalues of the transfer matrix. The two leading eigenvalues are $\Lambda_0 = (2\sinh 2J)^{N/2} \exp(\frac{1}{2}\sum_{r=0}^{N-1} \gamma_{2r+1})$ and $\Lambda_1 = (2\sinh 2J)^{N/2} \exp(\frac{1}{2}\sum_{r=1}^{N} \gamma_{2r})$, where γ_k is implicitly given by $\cosh\gamma_k = \cosh 2J \coth 2J - \cos(k\pi/N)$. At the critical point J_c of the sq lattice Ising model, where $J_c = \frac{1}{2}\ln(1 + \sqrt{2})$, one then obtains $\gamma_k = 2\psi_{\text{sq}}(\frac{k\pi}{2N})$. Here

$$\psi_{\text{sq}}(x) = \ln(\sin x + \sqrt{1 + \sin^2 x}). \tag{6}$$

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Then the critical free energy f_N and critical spin-spin correlation length ξ_N of Eq. (5) can be written as

$$f_N = \frac{1}{2}\ln 2 + \frac{1}{2N}\sum_{r=0}^{N-1}\gamma_{2r+1}, \tag{7}$$

$$\xi_N^{-1} = \frac{1}{2}\sum_{r=0}^{N-1}(\gamma_{2r+1} - \gamma_{2r}). \tag{8}$$

It is readily seen from Eqs. (6)–(8) that ξ_N^{-1} and Nf_N have odd parity as a function of N^{-1}. Therefore, in the following expansions of ξ_N^{-1} and Nf_N as a function of N^{-1}, we keep only odd terms.

To write f_N and ξ_N^{-1} in the form of Eqs. (1) and (2), we must evaluate Eqs. (7) and (8) asymptotically. These sums can be handled by using the Euler-Maclaurin summation formula [14]. After a straightforward calculation, we have

$$N(f_N - f_\infty) = \sum_{k=1}^{\infty}\frac{2B_{2k}}{(2k)!}(2^{2k-1} - 1)\left(\frac{\pi}{2N}\right)^{2k-1}\psi_{\rm sq}^{(2k-1)} = \frac{\pi}{12N} + \frac{7}{180}\left(\frac{\pi}{2N}\right)^3 + \frac{31}{756}\left(\frac{\pi}{2N}\right)^5 + \frac{10\,033}{75\,600}\left(\frac{\pi}{2N}\right)^7 + \ldots, \tag{9}$$

$$\xi_N^{-1} = \sum_{k=1}^{\infty}\frac{2B_{2k}}{(2k)!}(2^{2k} - 1)\left(\frac{\pi}{2N}\right)^{2k-1}\psi_{\rm sq}^{(2k-1)} = \frac{\pi}{4N} + \frac{1}{12}\left(\frac{\pi}{2N}\right)^3 + \frac{1}{12}\left(\frac{\pi}{2N}\right)^5 + \frac{1343}{5040}\left(\frac{\pi}{2N}\right)^7 + \ldots. \tag{10}$$

Here B_{2k} are the Bernoulli numbers and $\psi_{\rm sq}^{(2k-1)} = (d^{2k-1}\psi_{\rm sq}(x)/dx^{2k-1})_{x=0}$; $b_2 = \pi^3/96$ has been computed previously by Derrida and de Seze [13].

For the Ising model on the honeycomb lattice, Husimi and Syozi [9] found that $\Lambda_0 = (2\sinh 2J)^N\exp(\gamma_1 + \gamma_3 + \cdots + \gamma_{N-1})$ and $\Lambda_1 = (2\sinh 2J)^N\exp(\frac{1}{2}\gamma_0 + \gamma_2 + \cdots + \gamma_{N-2} + \frac{1}{2}\gamma_N)$, where the γ_r are given by $\cosh\gamma_r = \cosh 2J\cosh 2J^* - \sin^2\frac{\pi r}{N} - \cos\frac{\pi r}{N}\times(\sinh^2 2J\sinh^2 2J^* - \sin^2\frac{\pi r}{N})^{1/2}$. Here J^* is defined by $(\cosh 2J - 1)(\cosh 2J^* - 1) = 1$, so that one has $J^* = J$ at the critical point $[J_c = \frac{1}{2}\ln(2 + \sqrt{3})]$ and one then obtains $\gamma_r = 2\psi_{\rm hc}(\frac{r\pi}{2N})$, where

$$\psi_{\rm hc}(x) = \ln\{A(x) + \sqrt{A^2(x) - 1}\}, \tag{11}$$

with $A(x) = (\sqrt{8 + \cos^2 2x} - \cos 2x)/2$. Using the Euler-Maclaurin summation formula, we can write the free energy f_N and the inverse spin-spin correlation length ξ_N^{-1} for the hc lattice as

$$N(f_N - f_\infty) = \sum_{k=1}^{\infty}\frac{\sqrt{3}\,B_{2k}(2^{2k-1} - 1)}{(2k)!}\left(\frac{\pi}{2N}\right)^{2k-1}\psi_{\rm hc}^{(2k-1)} = \frac{\pi}{12N} - \frac{31}{210}\left(\frac{\pi}{3N}\right)^5 + \frac{511}{110}\left(\frac{\pi}{3N}\right)^9 + \ldots, \tag{12}$$

$$\xi_N^{-1} = \sum_{k=1}^{\infty}\frac{\sqrt{3}\,B_{2k}(2^{2k} - 1)}{(2k)!}\left(\frac{\pi}{2N}\right)^{2k-1}\psi_{\rm hc}^{(2k-1)} = \frac{\pi}{4N} - \frac{3}{10}\left(\frac{\pi}{3N}\right)^5 + \frac{93}{10}\left(\frac{\pi}{3N}\right)^9 + \ldots, \tag{13}$$

where $\psi_{\rm hc}^{(2k-1)} = (d^{2k-1}\psi_{\rm hc}(x)/dx^{2k-1})_{x=0}$.

For the case of the pt lattice, we note that one can use the star-triangle transformation to transform the hc to the pt lattice [11]. The amplitudes of the N^{-3} and N^{-7} correction terms are identically zero for the hc and the pt lattices.

The above results for the sq, hc, and pt lattices can be summarized as

$$N(f_N - f_\infty) = \sum_{k=1}^{\infty}\frac{2B_{2k}(2^{2k-1} - 1)}{\zeta(2k)!}\left(\frac{\pi}{2aN}\right)^{2k-1}\psi^{(2k-1)}, \tag{14}$$

$$\xi_N^{-1} = \sum_{k=1}^{\infty}\frac{2B_{2k}(2^{2k} - 1)}{\zeta(2k)!}\left(\frac{\pi}{2aN}\right)^{2k-1}\psi^{(2k-1)}, \tag{15}$$

where $a = 1$, $\zeta = 1$, $\psi(x) = \psi_{\rm sq}(x)$ for the sq lattice, $a = 1$, $\zeta = 2/\sqrt{3}$, $\psi(x) = \psi_{\rm hc}(x)$ for the hc lattice, and $a = 2$, $\zeta = 1/\sqrt{3}$, $\psi(x) = \psi_{tr}(x) = \psi_{\rm hc}(x)$ for the pt lattice. The ratios of the nonvanishing amplitudes of the $N^{-(2k-1)}$ correction terms in the spin-spin correlation length and the free energy expansion, i.e., b_k/a_k, are the same for all three lattices under consideration [15]. Thus we have established Eq. (3).

To check whether Eq. (3) is still valid for other models in the Ising universality class, we proceed to study a quantum spin model on a one-dimensional lattice of N sites with periodic boundary conditions, whose Hamiltonian is [10]

$$H = -\frac{\lambda}{2\gamma}\sum_{n=1}^{N}\sigma_n^z - \frac{1}{4\gamma}\sum_{n=1}^{N}[(1 + \gamma)\sigma_{n+1}^x\sigma_n^x + (1 - \gamma)\sigma_{n+1}^y\sigma_n^y], \tag{16}$$

where σ^x, σ^y, and σ^z are the Pauli matrices. The phase diagram is well known [16]. For all γ $(0 < \gamma \le 1)$, there is a critical point at $\lambda_c = 1$, which falls into the two-dimensional Ising universality class. The inverse correlation length ξ_i^{-1} is given by the difference in eigenvalues $E_i - E_0$ of the Hamiltonian H. In particular, the first energy gap gives the inverse spin-spin correlation length ξ_1^{-1} and the second energy gap is the inverse energy-energy correlation length ξ_2^{-1}. By expanding the exact solution of Eq. (16), Henkel [17] has obtained several finite-size correction terms to the ground state energy E_0, to the first $(E_1 - E_0)$ and second $(E_2 - E_0)$ energy gaps. We have extended the calculations to arbitrary order and found that

$$-E_0 - N\alpha_0 = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k-1} - 1)}{(2k)!}\left(\frac{\pi}{2N}\right)^{2k-1} \psi_q^{(2k-1)} = \frac{\pi}{12N} - \frac{7}{15}\left(\frac{1}{\gamma^2} - \frac{4}{3}\right)\left(\frac{\pi}{4N}\right)^3 - \frac{62}{63}\left(\frac{1}{\gamma^4} - \frac{16}{15}\right)\left(\frac{\pi}{4N}\right)^5 + \dots, \tag{17}$$

$$E_1 - E_0 = \sum_{k=1}^{\infty} \frac{2B_{2k}(2^{2k} - 1)}{(2k)!}\left(\frac{\pi}{2N}\right)^{2k-1} \psi_q^{(2k-1)} = \frac{\pi}{4N} - \left(\frac{1}{\gamma^2} - \frac{4}{3}\right)\left(\frac{\pi}{4N}\right)^3 - 2\left(\frac{1}{\gamma^4} - \frac{16}{15}\right)\left(\frac{\pi}{4N}\right)^5 + \dots, \tag{18}$$

$$E_2 - E_0 = \sum_{k=1}^{\infty} \frac{8k}{(2k)!}\left(\frac{\pi}{2N}\right)^{2k-1} \psi_q^{(2k-1)} = \frac{2\pi}{N} + 16\left(\frac{1}{\gamma^2} - \frac{4}{3}\right)\left(\frac{\pi}{4N}\right)^3 - 16\left(\frac{1}{\gamma^4} - \frac{16}{15}\right)\left(\frac{\pi}{4N}\right)^5 + \dots, \tag{19}$$

where $\psi_q^{(2k-1)} = (d^{2k-1}\psi_q(x)/dx^{2k-1})_{x=0}$, $\psi_q(x) = \sqrt{\sin^2 x - (1 - 1/\gamma^2)\sin^4 x}$, and α_0 is a nonuniversal number $\alpha_0 = \frac{2}{\pi}\int_0^{\pi} \psi_q(x)\,dx = 2[1 + \arccos\gamma/(\gamma\sqrt{1-\gamma^2})]/\pi$. Thus, the ratios of amplitudes for $(E_1 - E_0)$ and $(-E_0)$ also satisfy Eq. (3). Equations (17) and (19) also imply that the ratios $\bar{r}_k$ of amplitudes for $(E_2 - E_0)$ and $(-E_0)$ are γ independent and given by

$$\bar{r}_k = \frac{4k}{(2^{2k-1} - 1)B_{2k}}. \tag{20}$$

It is of interest to compare this finding with other results. The exact and numerical estimates [18] of the subdominant correction amplitudes for the sq, hc, and pt lattices are presented in Table I, which shows that the numerical values obtained by de Queiroz [18] are very close to our exact results. On the basis of conformal invariance, the asymptotic finite-size scaling behavior of the critical free energy and the inverse correlation length is found to be [19]

$$\lim_{N\to\infty} N^2(f_N - f_\infty) = \frac{c\pi}{6}, \tag{21}$$

$$\lim_{N\to\infty} N\xi_i^{-1} = \lim_{N\to\infty} N(E_i - E_0) = 2\pi x_i, \tag{22}$$

where c is the conformal anomaly number and x_i is the scaling dimension of the ith scaling field. For the 2D Ising model, we have $c = 1/2$, $x_1 = \eta/2 = 1/8$, and $x_2 = 1$, and the leading terms of Eqs. (14), (15), (17)–(19) for all of the sq, hc, and pt lattices and a quantum spin chain are consistent with Eqs. (21) and (22). Equations (21) and (22) imply immediately that their ratio is also universal, namely,

$$\lim_{N\to\infty} \frac{E_1 - E_0}{N(f_N - f_\infty)} = r_1 \quad \text{and} \quad \lim_{N\to\infty} \frac{E_2 - E_0}{N(f_N - f_\infty)} = \bar{r}_1, \tag{23}$$

where $r_1 = 12x_1/c$ and $\bar{r}_1 = 12x_2/c$. For the 2D Ising universality class we have $r_1 = 3$ and $\bar{r}_1 = 24$, which are consistent with Eqs. (3) and (20) for the case $k = 1$.

The corrections to Eqs. (21) and (22) can be calculated by means of a perturbated conformal field theory [20,21]. In general, any lattice Hamiltonian will contain correction terms to the critical Hamiltonian H_c,

$$H = H_c + \sum_p g_p \int_{-N/2}^{N/2} \phi_p(v)\,dv, \tag{24}$$

where g_p is a nonuniversal constant and $\phi_p(v)$ is a perturbative conformal field. Below we will consider the case with only one perturbative conformal field, say, $\phi_l(v)$. Then the eigenvalues of H are

$$E_n = E_{n,c} + g_l \int_{-N/2}^{N/2} \langle n|\phi_l(v)|n\rangle\,dv + \dots, \tag{25}$$

where $E_{n,c}$ are the critical eigenvalues of H. The matrix element $\langle n|\phi_l(v)|n\rangle$ can be computed in terms of the

TABLE I. Comparison of exact [Eqs. (9), (10), and (12)–(15)] and numerical [18] values for subdominant finite-size correction terms in free energy and inverse spin-spin correlation length expansion.

	Square		Honeycomb		Triangular	
	Exact	Numerical	Exact	Numerical	Exact	Numerical
a_2	0.15072...	0.150730(2)	0	$<10^{-6}$	0	$<10^{-6}$
b_2	0.322982...	0.322987(6)	0	$<10^{-8}$	0	$<10^{-8}$
a_3	0.39213...	0.385(1)	−0.18590...	−0.1865(10)	−0.01161...	−0.01165(5)
b_3	0.79692...	0.790(1)	−0.37780...	−0.3777(2)	−0.02361...	−0.02360(1)
a_4	3.13146...		0		0	
b_4	6.28759...		0		0	
a_5	48.9925...		7.03535...		0.02748...	
b_5	98.0809...		14.0844...		0.05501...	

universal structure constants (C_{nln}) of the operator product expansion [20]: $\langle n|\phi_l(v)|n\rangle = (2\pi/N)^{x_l} C_{nln}$, where x_l is the scaling dimension of the conformal field $\phi_l(v)$. The correlation lengths ($\xi_n^{-1} = E_n - E_0$) and the ground state energy (E_0) can be written as

$$\xi_n^{-1} = \frac{2\pi}{N} x_n + 2\pi g_l (C_{nln} - C_{0l0}) \left(\frac{2\pi}{N}\right)^{x_l - 1} + \dots, \tag{26}$$

$$E_0 = E_{0,c} + 2\pi g_l C_{0l0} \left(\frac{2\pi}{N}\right)^{x_l - 1} + \dots. \tag{27}$$

Equations (26) and (27) show that, while the amplitude of correction to scaling terms are not universal, ratios of them are. For the 2D Ising model, one finds [22] that the leading finite-size corrections ($1/N^3$) can be described by the Hamiltonian given by Eq. (24) with a single perturbative conformal field $\phi_l(v) = L_{-2}^2(v) + \bar{L}_{-2}^2(v)$ with scaling dimension $x_l = 4$. The universal structure constants C_{2l2}, C_{1l1}, and C_{0l0} can be obtained from the matrix element $\langle n|L_{-2}^2(v) + \bar{L}_{-2}^2(v)|n\rangle$, which have already been computed by Reinicke [23] ($C_{2l2} = 1729/5760$, $C_{1l1} = -7/720$, and $C_{0l0} = 49/5760$). Equations (26) and (27) imply that the ratios of first-order correction amplitudes for ($E_n - E_0$) and ($-E_0$) are universal and equal to $(C_{0l0} - C_{nln})/C_{0l0}$, which is consistent with Eqs. (3) and (20) for the cases $n = 1, k = 2$ $[(C_{0l0} - C_{1l1})/C_{0l0} = 15/7]$ and $n = 2, k = 2$ $[(C_{0l0} - C_{2l2})/C_{0l0} = -240/7]$, respectively. By comparing the amplitudes of the N^{-3} correction terms for the Ising model and the quantum spin chain with the general results of Eqs. (26) and (27), one can find that $g_l = (3/\gamma^2 - 4)/56\pi$ for the quantum spin chain and $g_l = -1/28\pi$ for the Ising model on the sq lattice. For the Ising model on the hc and pt lattices we find that $g_l = 0$, which indicates that at least two perturbative conformal fields are necessary to generate all finite-size correction terms. Further work has to be done to possibly evaluate exactly all finite-size correction terms from perturbative conformal field theory.

The results of this Letter inspire several problems for further studies: (i) On the basis of perturbated conformal field theory, can one find other universal amplitude ratios? (ii) How do such amplitudes behave in other models, for example, in the three-state Potts model? (iii) For the critical Ising model on a large $N \times M$ sq lattice (M/N is a finite number), we have obtained expansions in N^{-1} for the free energy, the internal energy, and the specific heat [24]. It is of interest to extend such expansions to inverse spin-spin correlation lengths and to hc and pt lattices, and to study whether the amplitude ratios are also universal.

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*Electronic address: huck@phys.sinica.edu.tw

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