Error Field Amplification and Rotation Damping in Tokamak Plasmas

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Toroidal rotation is normally very weakly damped in plasmas that are magnetically confined in the nominally toroidally symmetric tokamak. However, a strong damping of toroidal rotation is observed as such plasmas approach marginal stability for perturbations that produce a kinklike distortion of the plasma. It is shown that the damping of toroidal rotation by very small departures of the magnetic field from toroidal symmetry is greatly enhanced as marginal stability is approached. The response of a plasma to perturbations is studied using a set of electrical circuit elements, which provide an equation for the rotational damping that requires minimal information about the plasma.

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I. Introduction.—Toroidal rotation is weakly damped in plasmas that are confined by a magnetic field that has toroidal symmetry. Heating by the injection of beams of high-energy neutral particles can drive such plasmas to high velocities of toroidal rotation. Such phenomena are observed on the DIII-D tokamak, but the toroidal rotation is observed to be heavily damped as the plasma approaches stability limits for perturbations that produce a kinklike distortion of the plasma [1]. Here we show how very small toroidally asymmetric perturbations produced by the magnetic field coils (error fields) can be amplified by a plasma approaching marginal stability and produce a rapid damping of the toroidal rotation.

Angular momentum is transferred from the plasma to the coils that produce a field error by the off-diagonal part of the Maxwell stress tensor. Tokamak plasmas are essentially charge neutral, so the Maxwell stress tensor can be defined by $\vec{\nabla} \cdot \overleftrightarrow{T} \equiv \vec{j} \times \vec{B}$. The relevant off-diagonal part of the stress tensor is $T_{ij} = b_i b_j/\mu_0$, with b_i a vector component of the deviation of the magnetic field from toroidal symmetry. The transfer of the toroidal torque from the plasma to the coils is given by the integral $\tau_\varphi = (1/\mu_0) \oint b_\varphi \vec{b} \cdot d\vec{a}$ over any toroidally symmetric surface in the vacuum region between the plasma and the coils. Such surfaces can be defined by $\vec{x}_s(\theta, \varphi)$, with θ a poloidal and φ a toroidal angle. The toroidal component of the perturbation is $b_\varphi \equiv \vec{b} \cdot \partial \vec{x}_s/\partial \varphi$. The magnetic perturbation $\vec{b}$ is the sum of two fields: one due to currents in the plasma and the other due to currents in the coils producing the error. The torque transfer is a convolution integral of these two fields. A magnetic perturbation exerts a torque on a rotating plasma by many mechanisms. For example, the transfer of rotational energy to waves produces a torque on resonant surfaces. On these surfaces, the rotation frequency satisfies $\omega = k_\parallel C_w$, with $k_\parallel$ a wave number of the magnetic perturbation parallel to the unperturbed magnetic field, and C_w is the phase velocity of a wave (sound or shear Alfvén) along the field. The strength of the torque can be described by an empirically determined, dimensionless parameter α, which is defined by Eq. (4).

As a plasma approaches marginal stability for a kinklike perturbation, it becomes easier and easier to distort. At marginal stability, the amplitude of the plasma distortion due to a given external perturbation is inversely proportional to the torque parameter α as is the torque on plasma. This counterintuitive effect of a smaller torque parameter giving a larger torque is analogous to the freshman physics result that a light bulb with low resistance dissipates more energy than one with high resistance. We assume the plasma distortions are sufficiently small that the perturbations can be approximated by linear theory.

The interaction of an evolving plasma with an external magnetic perturbation involves three elements: (i) the external magnetic perturbation, (ii) the plasma, and (iii) the conducting structures surrounding the plasma. The conducting structures, such as chamber walls, are important if the time scale for plasma evolution is comparable to the time scale for currents to decay in these structures. Each of these elements can be represented by electrical circuit equations [2–4]. Sections (*II*) and (*III*) review the derivation of these circuit equations. Section (*IV*) applies these circuit equations to the amplification of an error field by a plasma approaching marginal stability and finds the enhanced damping of toroidal rotation produced by the error field.

II. Matrix circuit equations.—Currents in conducting structures, such as the chamber walls surrounding a plasma, can be calculated using a matrix circuit equation [2,4],

$$\overleftrightarrow{L} \cdot \frac{d\vec{I}}{dt} + \overleftrightarrow{R} \cdot \vec{I} = \vec{V}. \tag{1}$$

This equation is derived by multiplying $\vec{E} = -\partial \vec{A}/\partial t - \vec{\nabla}\phi$ by vector expansion functions $\vec{w}_i(\vec{x})$ with $\vec{\nabla} \cdot \vec{w}_i = 0$ and $\vec{j}(\vec{x}, t) = \sum I_i(t) \vec{w}_i(\vec{x})$. An integration over all of the space using

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{y}, t)}{|\vec{x} - \vec{y}|} d^3y$$

and $\vec{E} = \eta\vec{j}$ gives Eq. (1) with the inductance matrix defined by

$$L_{ij} \equiv \frac{\mu_0}{4\pi}\oint\frac{\vec{w}_1(\vec{x})\cdot\vec{w}_j(\vec{y})}{|\vec{x}-\vec{y}|}\,d^3x\,d^3y\,,$$

the resistance matrix defined by $R_{ij} = \int \eta\vec{w}_i(\vec{x})\cdot\vec{w}_j(\vec{x})\,d^3x$, and the voltage by $V_i = -\oint \phi\vec{w}_i\cdot d\vec{a}$. The voltage is zero unless an expansion function represents a circuit that is connected to a power supply.

The conducting structures surrounding the plasma are often thin compared to the wavelength of the perturbations. When this is true, the current in the structure can be approximated by a surface current, which means

$$\vec{w}_i(\vec{x}) = \vec{\nabla}\times\{f_i(\theta,\varphi)\delta[\vec{x}-\vec{x}_s(\theta,\varphi)]\hat{n}\},$$

with $\vec{x}_s(\theta,\varphi)$ defining the surface and $\hat{n}$ the surface normal. Another form for this equation may be easier to understand. In (r,θ,φ) coordinates in which $\vec{x}(r = r_s,\theta,\varphi) = \vec{x}_s(\theta,\varphi)$, one has $\delta(\vec{x}-\vec{x}_s)\hat{n} = \delta(r - r_s)\vec{\nabla}r$. The quantity $\kappa(\theta,\varphi,t) = \sum I_i(t)f_i(\theta,\varphi)$ is known as the current potential of the surface current.

III. Plasma circuit elements.—The circuit elements that represent the plasma are defined on a control surface, which is essentially the unperturbed, toroidally symmetric plasma surface, but technically lies infinitesimally outside of the plasma [3,4]. Any complete set of functions, $f_i(\theta,\varphi)$, integrated over that surface defines a column matrix that is called the perturbed plasma flux, $\Phi_i \equiv \oint f_i(\theta,\varphi)\vec{B}\cdot\hat{n}\,da$. Magnetically confined plasmas satisfy $\vec{\nabla}p = \vec{j}\times\vec{B}$, with p the plasma pressure, which implies $\vec{B}\cdot\hat{n}$ is zero in the absence of a perturbation. Outside of the plasma, the magnetic field due to the perturbed plasma current is equivalent to the field that would be produced by a unique surface current on the plasma surface. The column matrix $\vec{I}_p$ is the set of expansion coefficients for the current potential for this surface current expanded in the functions $f_i(\theta,\varphi)$. The plasma flux can then be written as $\vec{\Phi} = \overleftrightarrow{L}_p\cdot\vec{I}_p + \vec{\Phi}_x$, with $\overleftrightarrow{L}_p$ inductance of plasma surface and $\vec{\Phi}_x$ the flux due to currents external to plasma.

In toroidally symmetric plasmas, the expansion functions for perturbations can be written as $f_i(\theta,\varphi) = F_i(\theta)e^{iN\varphi}$, with N the toroidal mode number. The complex notation allows a simple representation of the toroidal phase shifts induced by the torque exerted by a rotating plasma on externally produced perturbations.

Since the perturbations to the plasma are assumed to be in a linear regime, the surface current equivalent of the perturbed plasma current has the form $I_p(t) = \int\overleftrightarrow{\rho}_g(t,\tau)\cdot\vec{\Phi}_x(t-\tau)\,d\tau$. If the plasma responds rapidly, one can approximate the response as instantaneous, $\vec{I}_p = \overleftrightarrow{\rho}\cdot\vec{\Phi}_x$. When the response is instantaneous, the energy and the torque that would be needed to drive an arbitrary surface current $\vec{J}$ in a conductor on the plasma surface determine [3,4] the reluctance $\overleftrightarrow{\rho}$. For such a current, $\vec{\Phi}_x = \overleftrightarrow{L}_p\cdot\vec{J}$. The power required to drive $\vec{J}$ is

$$\frac{d(\delta W)}{dt} = -\int\vec{j}\cdot\vec{E}\,d^3x = \frac{1}{4}\frac{d}{dt}(\vec{J}^\dagger\cdot\vec{\Phi}^\dagger + \vec{\Phi}^\dagger\cdot\vec{J}),$$

with $\vec{J}^\dagger$ the Hermitian conjugate of $\vec{J}$, and the torque required is

$$\tau_\varphi = -\int(\vec{j}\times\vec{B})\cdot\frac{\partial\vec{x}}{\partial\varphi}\,d^3x = i\frac{N}{2}(\vec{J}^\dagger\cdot\vec{\Phi} - \vec{\Phi}^\dagger\cdot\vec{J}).$$

These two relations imply the reluctance matrix is

$$\overleftrightarrow{\rho} = \overleftrightarrow{L}_p^{-1}\cdot(\overleftrightarrow{\Lambda} - \overleftrightarrow{L}_p)\cdot\overleftrightarrow{L}_p^{-1}, \tag{2}$$

with the matrix $\overleftrightarrow{\Lambda}$ defined by

$$\vec{\Phi}^\dagger\cdot\overleftrightarrow{\Lambda}^{-1}\cdot\vec{\Phi} = 2\delta W + \frac{i}{N}\tau_\varphi\,. \tag{3}$$

When the plasma can be approximated as ideal (no dissipation), the quantity $\delta W = \frac{1}{2}\vec{\Phi}^\dagger\cdot\overleftrightarrow{W}\cdot\vec{\Phi}$ can be calculated in diagonal form using a stability code such as [5] DCON. A stability code gives (i) the shape of each eigenvector, or mode, of $\overleftrightarrow{W}$, which is an $f_i(\theta,\varphi)$ on the plasma surface, and (ii) the eigenvalue W_i of $\overleftrightarrow{W}$, which is twice the energy of the mode. Equation (3) generalizes the well-known Hermitian $\overleftrightarrow{W}$ matrix of δW stability theory to a general interaction matrix $\overleftrightarrow{\Lambda}^{-1}$, which has both Hermitian (the energy) and anti-Hermitian (the toroidal torque) parts.

IV. Single mode model of field errors.—The general circuit equations that have been derived can be applied to the interaction of an evolving plasma with an externally produced field error. In the simplest approximation, one retains only the least stable plasma mode, which is called the single mode model.

In the single mode model, one needs the shape of the least stable plasma mode on the plasma surface $f_u(\theta,\varphi)$, the required energy to drive the mode, and the required torque. The energy of the single mode can be parametrized by $-s$, the energy required to drive the mode divided by the energy that would be required to drive the same normal field perturbation on the plasma surface without the plasma being present. The minus sign is inserted so s is positive for unstable perturbations. The torque between the mode and the plasma can be parametrized by the dimensionless coefficient α, which is torque divided by N times the energy required to drive the same normal field perturbation on the plasma surface without the plasma being present. The toroidal torque is

$$\tau_\varphi = N\alpha|\Phi|^2/L_p\,. \tag{4}$$

The coefficient α can be determined empirically. Dissipation coefficients, such as α, usually differ in plasmas from their simple theoretical estimates, so an empirical value for α may be preferable to such an estimate. The parameter s can be found using the DCON code [5]. Using the definitions of s and α, Eqs. (2) and (3) imply that the plasma reluctance is

$$\rho = -\left(\frac{1}{s - i\alpha} + 1\right)\frac{1}{L_p}.$$

Without rotation, $\alpha = 0$, the reluctance has a $1/s$ singularity, which represents the ease with which the plasma can be distorted as it approaches marginal stability, $s = 0$. The single mode model is generally an accurate approximation if $|s| \ll 1$ only for the least stable mode and if $\alpha \ll 1$, which is required for the validity of the ideal-plasma approximation made in the DCON code.

The magnetic flux through the wall in the single mode of the model is $\Phi_w = L_w I_w + M_{wp} I_p + \Phi_e^{(w)}$, while the perturbed plasma current is $I_p = \rho \Phi_x$ or

$$I_p = -\frac{1}{L_p}\left(\frac{1}{s - i\alpha} + 1\right)(M_{pw} I_w + \Phi_e^{(p)}).$$

The relation between the externally produced error field on plasma and on the wall is $\Phi_e^{(p)} = (M_{pw}/L_w)\Phi_e^{(w)}$. The interaction of the wall and the plasma is given by a coupling coefficient $c \equiv M_{wp} M_{pw}/(L_p L_w)$. In a simple cylindrical model, this coefficient would be given by the ratio of the plasma to the wall radius a/b and the poloidal mode number m with $c = (a/b)^{2m}$. A typical coupling coefficient for the least stable mode in DIII-D is $c \approx 0.2$.

The flux on the wall can be written in terms of the current in the wall and the externally driven error field as

$$\Phi_w = (1 - c)\left(1 - \frac{\frac{c}{1-c}}{s - i\alpha}\right)(L_w I_w + \Phi_e^{(w)}).$$

The circuit equation $d\Phi_w/dt = -R_w I_w$ implies, with $\gamma_w = R_w/L_w$, that

$$\frac{d\Phi_w}{dt} = -(v_g + i\omega_r)\Phi_w + \gamma_w \Phi_e^{(w)}.$$

The growth rate of the perturbation, which is called the resistive wall mode growth rate, is

$$v_g = \frac{s(\frac{c}{1-c} - s) - \alpha^2}{(\frac{c}{1-c} - s)^2 + \alpha^2}\,\frac{\gamma_w}{1 - c},$$

and the equilibrium rotation rate of the resistive wall mode is

$$\omega_r = \frac{\frac{c}{1-c}\,\alpha}{(\frac{c}{1-c} - s)^2 + \alpha^2}\,\frac{\gamma_w}{1 - c}.$$

The perturbation on the plasma is proportional to the perturbation on the wall,

$$\Phi = \frac{1}{1 - c}\,\frac{1}{\frac{c}{1-c} - (s - i\alpha)}\,\frac{M_{pw}}{L_w}\,\Phi_w.$$

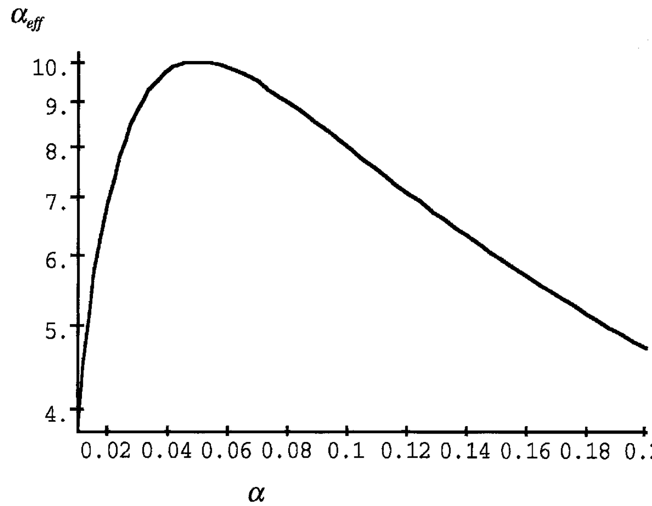

FIG. 1. Enhancement of the effective toroidal rotation damping coefficient, α_{eff}, from its value in the absence of plasma amplification effects, α, near marginal stability for a plasma kink mode, $s = -0.05$.

The equations that we have derived can be integrated for a plasma becoming increasingly less stable, which means s is negative but approaching zero. If the plasma evolution is slow compared to $1/v_g$, the results are accurately approximated by the steady-state answers. In steady state, the perturbation on the plasma is $\Phi = -\Phi_e^{(p)}/(s - i\alpha)$ and the torque is $\tau_\varphi = N\alpha_{\text{eff}}(s)|\Phi_e^{(p)}|^2/L_p$ with the effective torque coefficient $\alpha_{\text{eff}} \equiv \alpha/(s^2 + \alpha^2)$.

When the stability parameter is small but negative, the rate of momentum damping is greatly enhanced over its nominal value, $\alpha_{\text{eff}} = \alpha$. Figure 1, which plots α_{eff} versus α for $s = -0.05$, illustrates this point. At exact marginal stability, $s = 0$, the amplitude of the plasma distortion is proportional to $1/\alpha$, so the torque, which is proportional to the distortion squared times α, is also proportional to $1/\alpha$.

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[1] A. M. Garofalo, A. D. Turnbull, M. E. Austin, J. Bialek, M. S. Chu, K. J. Comer, E. D. Fredrickson, R. J. Groebner, R. J. La Haye, L. L. Lao, E. A. Lazarus, G. A. Navratil, T. H. Osborne, B. W. Rice, S. A. Sabbagh, J. T. Scoville, E. J. Strait, and T. S. Taylor, Phys. Rev. Lett. **82**, 3811 (1999).

[2] A. H. Boozer, J. Bialek, and G. A. Navratil (to be published).

[3] J. Bialek, A. H. Boozer, M. E. Mauel, and G. A. Navratil, Phys. Plasmas (to be published).

[4] A. H. Boozer, Phys. Plasmas **6**, 3180 (1999).

[5] A. H. Glasser and M. S. Chance, Bull. Am. Phys. Soc. **42**, 1848 (1997), abstract dMopP12.