The Quark Condensate from K_{e_4} Decays

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We show that, independently of the size of the quark condensate, chiral symmetry correlates the two S-wave $\pi\pi$ scattering lengths. In view of this constraint, the new precision data on K_{e_4} decay allow a remarkably accurate determination of these quantities. The result confirms the hypothesis that the quark condensate is the leading order parameter.

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Since the masses of the two lightest quarks are very small, the Hamiltonian of QCD is almost exactly invariant under the group $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ of chiral rotations. On phenomenological grounds, it is known that this symmetry is spontaneously broken, the pions playing the role of the corresponding Goldstone bosons [1]. If the symmetry were exact, the pions would be massless. According to Gell-Mann, Oakes, and Renner [2], the square of the pion mass is proportional to the product of the quark masses and the quark condensate:

$$M_\pi^2 \simeq \frac{1}{F_\pi^2} \times (m_u + m_d) \times |\langle 0|\bar{u}u|0\rangle| \,. \qquad (1)$$

The factor of proportionality is given by the pion decay constant F_π. The term $m_u + m_d$ measures the explicit breaking of chiral symmetry, while the quark condensate $\langle 0|\bar{u}u|0\rangle$ is a measure of the spontaneous symmetry breaking: It may be viewed as an order parameter and plays a role analogous to the spontaneous magnetization of a magnet.

The approximate validity of the relation (1) was put to question by Stern and collaborators [3], who pointed out that there is no experimental evidence for the quark condensate to be different from zero. Indeed, the dynamics of the ground state of QCD is not understood—it could resemble the one of an antiferromagnet, where, for dynamical reasons, the most natural candidate for an order parameter, the magnetization, happens to vanish. What can be shown from first principles is only that (i) the expansion of M_π^2 in powers of the quark masses starts with a linear term,

$$\begin{aligned} M_\pi^2 &= M^2 - \frac{\bar{\ell}_3}{32\pi^2 F^2} M^4 + O(M^6) \,, \\ M^2 &= (m_u + m_d) B \,, \end{aligned} \qquad (2)$$

and (ii) the coefficient B of the linear term is given by the value of $|\langle 0|\bar{u}u|0\rangle|/F_\pi^2$ in the limit $m_u, m_d \to 0$. The quantity $\bar{\ell}_3$ is one of the coupling constants occurring in the effective Lagrangian at order p^4. The symmetry does not determine its size. The crude estimates underlying the standard version of chiral perturbation theory (CHPT) [4] indicate values in the range $0 < \bar{\ell}_3 < 5$. The term of order M^4 is then very small compared to the one of order M^2, so that the Gell-Mann-Oakes-Renner formula is obeyed very well. Stern and collaborators investigate the more general framework, referred to as generalized CHPT, where arbitrarily large values of $\bar{\ell}_3$ are considered. The quartic term in Eq. (2) can then take values comparable to the "leading," quadratic one. If so, the dependence of M_π^2 on the quark masses would fail to be approximately linear, even for values of m_u and m_d of the order of a few MeV. A different bookkeeping for the terms occurring in the chiral perturbation series is then needed [3]—the standard chiral power counting is adequate only if $\bar{\ell}_3$ is not too large.

The purpose of the present note is to show that (i) in the generalized scenario, the low energy structure is controlled by a single parameter, here denoted by $\bar{\ell}_3$, (ii) this parameter can be determined on the basis of the K_{e_4} data taken recently at Brookhaven [5,6], and (iii) the result beautifully confirms the Gell-Mann-Oakes-Renner formula.

The following analysis relies on the fact that the low energy properties of the pions are controlled by two parameters: the S-wave scattering lengths a_0^0, a_0^2. If these are given, the Roy equations [7] allow us to calculate the scattering amplitude in terms of the absorptive parts above 800 MeV and the available experimental information about the latter suffices to evaluate the relevant dispersion integrals, to within small uncertainties [8].

Weinberg's low energy theorem [9] predicts the two scattering lengths in terms of the pion decay constant, so that the scattering amplitude is then fully determined. The prediction is of limited accuracy, because it holds only to leading order of an expansion in powers of the quark masses m_u and m_d. At first nonleading order of the expansion in powers of momenta and quark masses, the scattering amplitude can be expressed in terms of F_π, M_π, and the coupling constants $\ell_1, \ldots, \ell_4$ that occur in the derivative expansion of the effective Lagrangian at order p^4 (throughout, we ignore isospin breaking effects and work with $m_u = m_d = m$). The terms ℓ_1 and ℓ_2 manifest themselves in the energy dependence of the scattering amplitude and can thus be determined phenomenologically. The

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term ℓ_3 was mentioned above—the range considered for this coupling constant makes the difference between standard and generalized CHPT. Finally, ℓ_4 is related to the slope of the scalar form factor, which is known rather accurately from dispersion theory: $\langle r^2\rangle_s = 0.61 \pm 0.04$ fm^2 [10–12].

As pointed out long ago [13], there is a low energy theorem that relates the S-wave scattering lengths to the scalar radius:

$$2a_0^0 - 5a_0^2 = \frac{3M_\pi^2}{4\pi F_\pi^2}\left\{1 + \frac{1}{3}M_\pi^2\langle r^2\rangle_s + \frac{41M_\pi^2}{192\pi^2F_\pi^2}\right\} + O(m^3)\,. \tag{3}$$

The theorem shows that the first order correction to the Weinberg formula for this particular combination of scattering lengths is determined by $\langle r^2\rangle_s$. It correlates the two scattering lengths, irrespectively of the numerical value of ℓ_3: The correlation holds both in standard and generalized CHPT.

The corrections occurring in Eq. (3) at order m^3 are also known [14]. In the following, we analyze the correlation at that level of precision, using the method described in Ref. [10]—except that we now treat the coupling constant ℓ_3 as a free parameter. For the symmetry breaking couplings entering the effective Lagrangian at order p^6, we assume that the estimates given in Ref. [15] are valid within a factor of 2 (see [16] for a detailed error analysis). The experimental input used for the Roy equations is taken from Ref. [8]. Up to the noise attached to these ingredients, the Roy equations then determine the scattering amplitude as a function of the parameter ℓ_3. In particular, we may calculate a_0^0 and a_0^2 as functions of ℓ_3. The result is well described by a parabola:

$$\begin{aligned} a_0^0 &= 0.225 - 1.6\times10^{-3}\bar\ell_3 \\ &\quad - 1.3\times10^{-5}(\bar\ell_3)^2, \\ a_0^2 &= -0.0434 - 3.6\times10^{-4}\bar\ell_3 \\ &\quad - 4.3\times10^{-6}(\bar\ell_3)^2. \end{aligned} \tag{4}$$

Eliminating the parameter ℓ_3, we obtain the following correlation between a_0^2 and a_0^0:

$$a_0^2 = -0.0444 \pm 0.0008 + 0.236(a_0^0 - 0.22) - 0.61(a_0^0 - 0.22)^2 - 9.9(a_0^0 - 0.22)^3. \tag{5}$$

The error given accounts for the various sources of uncertainty in our input. The relation is indicated in Fig. 1: The values of a_0^0 and a_0^2 are constrained to a narrow strip that runs along the lower edge of the universal band, which is indicated by the tilted straight lines. As discussed in Ref. [8], a qualitatively similar correlation also results from the Olsson sum rule [18]—the two conditions are perfectly compatible, but the one above is considerably more stringent.

The analysis of the final state distribution observed in the decay $K \to \pi\pi e\nu$ yields a measurement of the phase difference $\delta(s) \equiv \delta_0^0(s) - \delta_1^1(s)$, for $4M_\pi^2 < s < M_K^2$. At those energies, $\delta(s)$ is dominated by the contribution $\propto a_0^0$ from the S-wave scattering length. The correlation between a_0^2 and a_0^0 allows us to correct for the higher order terms of the threshold expansion and to express the phase difference in terms of a_0^0 and q, where q is the c.m. momentum in units of M_π, $s = 4M_\pi^2(1+q^2)$. In the region of interest ($q < 1$, $0.18 < a_0^0 < 0.26$), the prediction reads

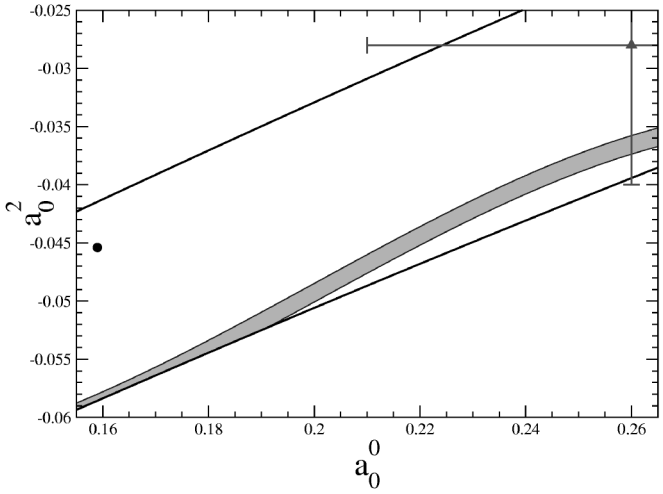

FIG. 1. S-wave scattering lengths. The Roy equations only admit solutions in the "universal band," spanned by the two tilted lines. The correlation in Eq. (5) constrains the values to the narrow shaded strip. The full circle indicates Weinberg's leading order result, while the triangle with error bars shows the phenomenological range permitted by the old data, $a_0^0 = 0.26 \pm 0.05$, $a_0^2 = -0.028 \pm 0.012$ [17].

$$\begin{aligned} \delta_0^0 - \delta_1^1 &= \frac{q}{\sqrt{1+q^2}}(a_0^0 + q^2 b + q^4 c + q^6 d) \pm e\,, \\ b &= 0.2527 + 0.151\Delta a_0^0 \\ &\quad + 1.14(\Delta a_0^0)^2 + 35.5(\Delta a_0^0)^3, \\ c &= 0.0063 - 0.145\Delta a_0^0, \qquad d = -0.0096\,, \end{aligned} \tag{6}$$

with $\Delta a_0^0 = a_0^0 - 0.22$. The uncertainty in this relation mainly stems from the experimental input used in the Roy equations and is not sensitive to a_0^0:

$$e = 0.0035q^3 + 0.0015q^5. \tag{7}$$

The prediction (6) is illustrated in Fig. 2, where the energy dependence of the phase difference is shown for $a_0^0 = 0.18$, 0.22, and 0.26. The width of the corresponding bands indicates the uncertainties, which according to (7) grow in proportion to q^3—in the range shown, they amount to less than a third of a degree.

The figure shows that the data of Ref. [19] barely distinguish between the three values of a_0^0 shown. The preliminary results of the E865 experiment at Brookhaven [6] are significantly more precise, however. The best fit to these data is obtained for $a_0^0 = 0.218$, with $\chi^2 = 5.7$ for 5 degrees of freedom. This beautifully confirms the very sharp predictions obtained on the basis of standard CHPT: $a_0^0 = 0.220 \pm 0.005$, $a_0^2 = -0.0444 \pm 0.0010$ [10,20].

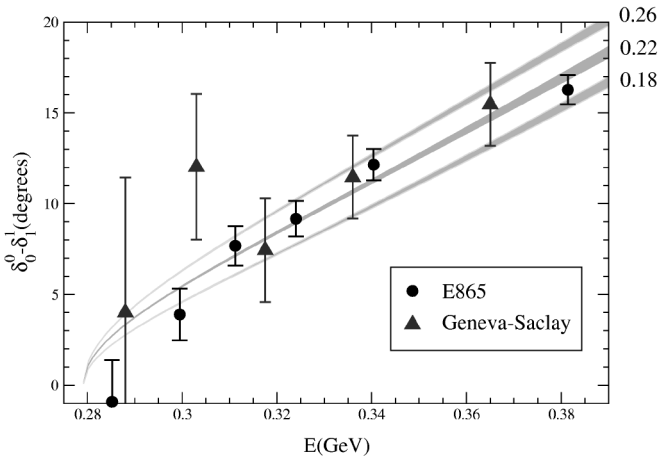

FIG. 2. Phase relevant for the decay $K \to \pi\pi e\nu$. The three bands correspond to the three indicated values of the S-wave scattering length a_0^0. The uncertainties are dominated by those from the experimental input used in the Roy equations. The triangles are the data points of Ref. [19], while the full circles represent the preliminary E865 results [6].

There is a marginal problem only with the bin of lowest energy: The corresponding scattering lengths are outside the region where the Roy equations admit solutions. In view of the experimental uncertainties attached to that point, this discrepancy is without significance: The difference between the central experimental value and the prediction amounts to $1\frac{1}{2}$ standard deviations. Note also that the old data are perfectly consistent with the new ones: The overall fit yields $a_0^0 = 0.221$ with $\chi^2 = 8.3$ for 10 degrees of freedom.

The relation (6) can be inverted, so that each one of the values found for the phase difference yields a measurement of the scattering length a_0^0. The result is shown in Fig. 3. The experimental errors are remarkably small. It is not unproblematic, however, to treat the data collected in the different bins as statistically independent: In the presence of correlations, this procedure underestimates the actual uncertainties. Also, since the phase difference rapidly rises with the energy, the binning procedure may introduce further uncertainties. To be on the conservative side, we estimate the uncertainties by using the 95% confidence limit, where we obtain $a_0^0 = 0.221 \pm 0.026$. For the final data analysis, we refer to a forthcoming paper by the E685 Collaboration.

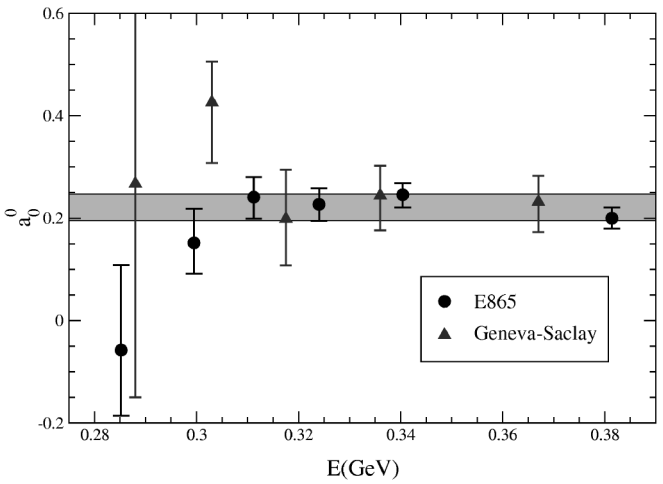

FIG. 3. K_{e4} data on the scattering length a_0^0. The triangles are the data points of Ref. [19], while the full circles represent the preliminary E865 results [6]. The horizontal band indicates the mean value, $a_0^0 = 0.221 \pm 0.026$.

We may translate the result into an estimate for the magnitude of the coupling constant $\bar{\ell}_3$. In this language, the above conclusion for the value of a_0^0 corresponds to $|\bar{\ell}_3| \lesssim 16$. Although this is a coarse estimate, it implies that the Gell-Mann-Oakes-Renner relation does represent a decent approximation: More than 94% of the pion mass stems from the first term in the quark mass expansion (2), i.e., from the term that originates in the quark condensate.

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