Supersymmetric Ratchets

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The overdamped Brownian motion in a periodic potential under far from equilibrium conditions is considered. A large class of systems with an intrinsic asymmetry, called supersymmetric ratchets, is identified for which the occurrence of directed transport can be ruled out without any fine-tuning of parameters.

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At thermal equilibrium, a Brownian particle in a spatially periodic, asymmetric "ratchet" potential cannot exhibit a systematic drift in one or the other preferential direction due to the second law of thermodynamics [1,2]. Away from equilibrium, the occurrence of a systematic particle current (ratchet effect) is observed generically, even though all acting forces still average out to zero [3–7]. While interesting exceptional cases with zero net motion (current reversals [8–12]) are still possible, they are untypical in the sense that they require a fine-tuning of certain model parameters. In other words, *away from thermal equilibrium, the absence rather than the presence of directed transport in spite of the broken spatial symmetry is the truly astonishing situation*. In our present work, an entire class of such exceptional cases is identified, which, in particular, do not require a fine-tuning of model parameters.

As a first model class, we consider the one-dimensional, overdamped "tilting-ratchet" dynamics [3–7]

$$\eta \dot{x}(t) = -V'(x(t)) + f(t) + \xi(t), \tag{1}$$

where η is the viscous friction coefficient, $V(x)$ is a ratchet potential with period L, and thermal fluctuations are modeled by Gaussian white noise $\xi(t)$ with zero average and correlation

$$\langle \xi(t)\xi(s)\rangle = 2\eta k_B T \delta(t - s). \tag{2}$$

The "tilting force" $f(t)$ drives the system away from thermal equilibrium, and may be either an unbiased periodic function of time or an unbiased stationary stochastic process. In other words, all forces on the right-hand side of (1) are zero after averaging over space, time, and statistical ensembles. The quantity of central interest is the average particle current in the long time limit

$$\langle \dot{x} \rangle := \lim_{t-t_0\to\infty} \frac{x(t) - x(t_0)}{t - t_0}, \tag{3}$$

which takes the same value for each realization $x(t)$ with probability one (self-averaging).

As an *example* and a motivation for our systematic discussion below, we consider a ratchet potential $V(x)$ as depicted in Fig. 1 and a nonequilibrium forcing $f(t)$ which may be for instance (i) a sinusoidal, time-periodic function, (ii) a symmetric dichotomous noise, (iii) an Ornstein-Uhlenbeck process, (iv) a symmetric white shot noise, or (v) an arbitrary linear combination thereof. The extensive previous investigations [3–7,9–12] of such ratchet systems (1) strongly suggest that in either of these examples a nonvanishing particle current (3) is to be generically expected. Surprisingly indeed, we will demonstrate below that the contrary is the case: The particle current identically vanishes for *any* choice of the friction coefficient η, the temperature T, and the characteristic amplitude, time scale, etc. of the driving $f(t)$. For additional examples, see also Fig. 2.

We begin our systematic discussion with the following definitions: We call a *potential* $V(x)$ *supersymmetric* if there exist a Δx and a ΔV such that $-V(x) = V(x + \Delta x) + \Delta V$ for all x. Since additive constants are irrelevant for the potential $V(x)$, we henceforth focus on the case $\Delta V = 0$. Further, by applying the above defined supersymmetry transformation twice, we can conclude that $V(x + 2\Delta x) = V(x)$ for all x. Under the assumption that L is the fundamental period of $V(x)$, i.e., the smallest $z > 0$ with $V(x + z) = V(x)$, we can conclude that $\Delta x = L/2$. The *supersymmetry criterion* thus takes the form

$$-V(x) = V(x + L/2). \tag{4}$$

An immediate implication of (4) is that for any minimum of $V(x)$, say at $x = x_{\min}$, there exists a corresponding

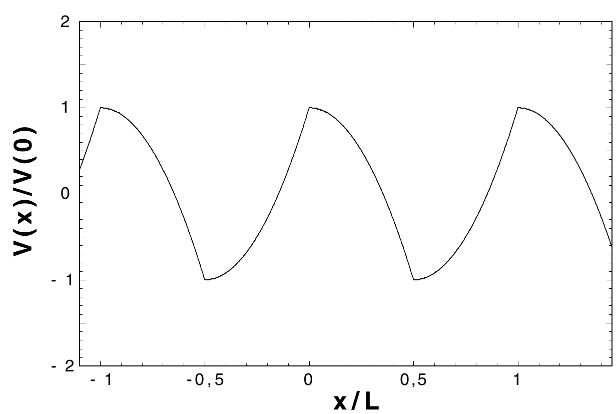

FIG. 1. Example of a ratchet potential $V(x)$, satisfying the supersymmetry condition (4).

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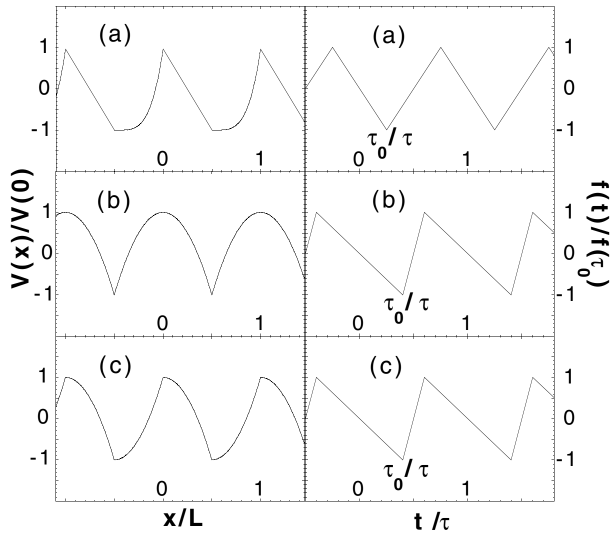

FIG. 2. Three combinations of L-periodic potentials $V(x)$ and τ-periodic drivings $f(t)$ with different types of symmetries. The reader is invited to guess in which cases a vanishing current (3) arises in the stochastic dynamics (1) and (2), and in which cases a finite current occurs generically (without fine-tuning of parameters). For the resolution, see main text.

maximum at $x = x_{\min} + L/2$ and vice versa. For the rest, the condition (4) is still satisfied by a very large class of potentials $V(x)$ of the general form

$$V(x) = \sum_{n=1,3,5,\ldots}^{\infty} a_n \cos\left(\frac{2\pi n x}{L} + \phi_n\right). \tag{5}$$

A typical example is depicted in Fig. 1.

Turning to the *driving* $f(t)$, we will call it *supersymmetric* if for a *periodic* $f(t)$ we have that $-f(t) = f(-t + \Delta t)$ for all t and an appropriate Δt, which can be transformed to zero by an irrelevant shift of the time scale. For a *stochastic* $f(t)$ (unbiased and stationary) we speak of supersymmetry if all statistical properties of the process $-f(t)$ are identical to those of $f(-t)$. By extending the meaning of the equality sign "=" along this statistical spirit, the *supersymmetry criterion* can thus be written for both, periodic and stochastic $f(t)$, as

$$-f(t) = f(-t). \tag{6}$$

As a consequence, a supersymmetric, *time periodic* $f(t)$ with period τ is of the general form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{\tau}\right), \tag{7}$$

while for a *stochastic* $f(t)$ the condition (6) is equivalent to the requirement [13] that

$$\langle f(t_1) f(t_2) \cdots f(t_n)\rangle = (-1)^n \langle f(-t_1) f(-t_2) \cdots f(-t_n)\rangle \tag{8}$$

for all integers $n \geq 1$ and all times $t_1, t_2, \ldots, t_n$. For examples, see (i)–(v) above. Especially, linear combinations of periodic and stochastic $f(t)$ are admissible as well.

Regarding the above introduced notion of supersymmetry, we remark that for undriven [$f(t) \equiv 0$] systems (1) the connection with supersymmetric quantum mechanics [14] is due to [15]. The basic idea is to transform the Fokker-Planck equation [16] associated with (1) and (2) into a Schrödinger-type equation [17]. By replacing in this equation the potential by its supersymmetric partner potential (in the quantum mechanical sense) a new Schrödinger equation emerges which can be transformed back into a new Fokker-Planck equation. The potentials of the original and the new Fokker-Planck equations then coincide (up to irrelevant shifts Δx and ΔV of the origin) if and only if the supersymmetry condition (4) is satisfied. In the presence of a driving $f(t)$ in (1), a similar line of reasoning has been developed in [18], yielding the supersymmetry condition (6). *Throughout our present paper, we will borrow the previously established notion of supersymmetry for the conditions (4) and (6), but we will neither exploit nor further discuss their connection with quantum mechanical concepts.*

We now come to the central point of our paper, namely the proof that *supersymmetry of both $V(x)$ and $f(t)$ implies $\langle \dot{x}\rangle = 0$*: Introducing $z(t) := x(-t) + L/2$, we can infer $\dot{z}(t) = -\dot{x}(-t)$, i.e., the time averaged currents (3) satisfy $\langle \dot{z}\rangle = -\langle \dot{x}\rangle$. In doing so, we have exploited the fact that only deterministic and/or *stationary* stochastic processes appear in (1), hence the evolution of the dynamics (1) backward in time does not give rise to any problem. Especially, $-\xi(-t)$ is statistically equivalent to the forward Gaussian white noise $\xi(t)$. On the other hand, if both $V(x)$ and $f(t)$ are supersymmetric according to (4) and (6) then one can readily see that $z(t)$ satisfies the same dynamics (1) as $x(t)$. Because of the self-averaging property of the current in (3) it follows that $\langle \dot{z}\rangle = \langle \dot{x}\rangle$. In view of our previous finding $\langle \dot{z}\rangle = -\langle \dot{x}\rangle$ we can conclude that $\langle \dot{x}\rangle = 0$; see also [19].

It is instructive to compare this result with the corresponding situation in "symmetric" instead of supersymmetric systems. To this end, a *potential* $V(x)$ will be called *symmetric* if there exists a Δx such that $V(-x) = V(x + \Delta x)$, or, after an irrelevant shift of the x scale,

$$V(-x) = V(x). \tag{9}$$

Thus, a symmetric $V(x)$ can be recast into the general form

$$V(x) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{2\pi n x}{L}\right). \tag{10}$$

Further, a *periodic* driving $f(t) = f(t + \tau)$ is called *symmetric* if there exists a Δt such that $-f(t) = f(t + \Delta t)$. As in (4) one can infer [11] that $\Delta t = \tau/2$, i.e.,

$$-f(t) = f(t + \tau/2), \tag{11}$$

or, equivalently, that $f(t)$ is of the general form

$$f(t) = \sum_{n=1,3,5,\ldots}^{\infty} d_n \cos\left(\frac{2\pi n t}{\tau} + \psi_n\right). \tag{12}$$

Finally, if $f(t)$ is a stationary *stochastic* process, then we call it *symmetric* if all statistical properties of $-f(t)$ are identical to those of $f(t)$. In the same spirit as in (6) this may be symbolically indicated as

$$-f(t) \doteq f(t). \tag{13}$$

Equivalent to this condition is the requirement [cf. (6), (8)] that all odd moments of $f(t)$ vanish [20], i.e.,

$$\langle f(t_1)f(t_2)\cdots f(t_{2n+1})\rangle = 0 \tag{14}$$

for all integers $n \geq 0$ and all times $t_1, t_2, \ldots, t_{2n+1}$. By considering $z(t) := -x(t + \tau/2)$ one finds along the same line of reasoning as in the preceding paragraph that *symmetry of both* $V(x)$ *and* $f(t)$ *implies* $\langle \dot{x} \rangle = 0$.

We emphasize again that the conclusion $\langle \dot{x} \rangle = 0$ holds true only if either *both* the potential *and* the driving are symmetric, or if *both* of them are supersymmetric. In any other case, $\langle \dot{x} \rangle \neq 0$ is expected generically (without fine-tuning of parameters). As an illustration we consider the three *examples from Fig. 2*. In order to bring out the essential features as clearly as possible, we have chosen stylized, nonsmooth potentials $V(x)$ and drivings $f(t)$ in Fig. 2 and we restrict ourselves to time periodic $f(t)$. More general examples can be easily constructed.

Figure 2a illustrates the most common case [3–7,9–12], namely, an asymmetric (but not supersymmetric) ratchet potential $V(x)$ in combination with a symmetric (and at the same time supersymmetric) $f(t)$. The potential is somewhat special in the sense that the distance between subsequent extrema is—as for supersymmetric potentials—always equal to $L/2$. Nevertheless, $\langle \dot{x} \rangle \neq 0$ is expected generically, i.e., without fine-tuning the model parameters η, T, τ, $f(\tau/4)$, L, and $V(0)$ in (1) and (2). We have corroborated this and all the following such "expectations" by numerical simulations of (1).

Figure 2b depicts a second prototypical scenario, namely, a symmetric (but not supersymmetric) potential $V(x)$ (i.e., *not* a ratchet potential) in combination with a supersymmetric driving $f(t)$. The matching of the linear pieces of this driving $f(t)$ is characterized by the parameter $\tau_0 \in (0, \tau/2)$, giving rise to a symmetric driving for $\tau_0 = \tau/4$ (see Fig. 2a) and an asymmetric driving in any other case (see Fig. 2b). As a result, a generically nonzero current is obtained unless $\tau_0 = \tau/4$; see also [11,21]. Qualitatively, the same findings are recovered for a so-called *harmonic mixing* signal [22] of the form $f(t) = A\sin(\omega t) + B\sin(2\omega t + \phi)$ when $\phi = 0$, corresponding to a supersymmetric but (for $A, B \neq 0$) asymmetric driving; see (7) and (11). In either case, if the potential $V(x)$ in Fig. 2b is replaced by a pure cosine potential, then both $V(x)$ and $f(t)$ are supersymmetric [see (5)] and hence the current must vanish. By further deforming the potential $V(x)$ such that it becomes asymmetric but does not leave the class of supersymmetric potentials [see (5) and Fig. 2c], the same result $\langle \dot{x} \rangle = 0$ subsists. Finally, one can deform the driving $f(t)$ into a pure sinusoidal shape, leading us back to the setup which we used as a motivation at the beginning of our paper.

Next we turn to a second main class of models—so-called "pulsating ratchets"—governed by the dynamics [4–7]

$$\eta \dot{x}(t) = -V'(x(t), f(t)) + \xi(t), \tag{15}$$

where $V'(x,f) := \partial V(x,f)/\partial x$. While $f(t)$ and $\xi(t)$ are assumed to have the same properties as for the tilting ratchet scheme (1) and (2), the potential $V(x,f)$ is now required to be L periodic in x for any fixed f value.

For a *time periodic* $f(t)$ one finds upon comparison of $z(t) := x(-t) + L/2$ with $x(t)$ along the very same line of reasoning as above that $\langle \dot{x} \rangle = 0$ provided there exist $\Delta x, \Delta t, \Delta V$ such that $-V(x, f(t)) = V(x + \Delta x, f(-t + \Delta t)) + \Delta V$ for all x and t. For a *stochastic* $f(t)$, the same conclusion follows provided all statistical properties of $-V(x, f(t))$ are identical to those of $V(x + \Delta x, f(-t)) + \Delta V$. As usual, we can and will choose the t and V origins such that $\Delta t = 0$ and $\Delta V = 0$, while Δx can be identified with $L/2$ as in (4). The *supersymmetry criterion* can thus be rewritten for both periodic and stochastic $f(t)$ as

$$-V(x, f(t)) \doteq V(x + L/2, f(-t)). \tag{16}$$

Likewise, upon comparison of $z(t) := -x(t + \tau/2)$ with $x(t)$ one finds that $\langle \dot{x} \rangle = 0$ on the condition that the *symmetry criterion*

$$V(-x, f(t)) \doteq V(x, f(t)) \tag{17}$$

is fulfilled. In summary, *if either of the criteria (16) or (17) is met, a vanishing current in (15) is the consequence, while in any other case, a nonvanishing current is generically expected.*

Within the realm of pulsating ratchets (15), one important subclass is so-called *fluctuating potential ratchets,* characterized by a potential of the form

$$V(x, f(t)) = V(x)[1 + f(t)]. \tag{18}$$

The summand 1 is a matter of convention, reflecting a kind of "unperturbed" contribution to the total potential. A special case is *on-off ratchets* when $f(t)$ can take only two possible values, one of them being -1 (potential "off"). In this case of a fluctuating potential (18), the supersymmetry condition for the total potential $V(x, f(t))$ in (16) is satisfied if and only if the static part $V(x)$ fulfills the corresponding supersymmetry criterion (4) and $f(t)$ is time inversion invariant [$f(-t) \doteq f(t)$]. *Examples* of this kind, for which the result $\langle \dot{x} \rangle = 0$ may be once more rather unexpected at first glance, are potentials $V(x)$ as in Fig. 1 in combination with a driving $f(t)$ of the type (i)–(v) from above. On the other hand, the symmetry criterion (17) with (18) is tantamount to the corresponding symmetry condition (9) for $V(x)$, independent of any further properties of $f(t)$.

Returning to general potentials of the form $V(x, f(t))$, the supersymmetry condition (16) is still satisfied by a very large class of such potentials and their exhaustive characterization on an intuitive level seems rather difficult.

Yet, two sufficient (but not necessary) simple conditions for (16) can be given, namely, (a) the potential is of the form

$$V(x, f(t)) = V_1(x) + V_2(x) f(t), \tag{19}$$

where $V_1(x)$ is supersymmetric according to (4), $V_2(x)$ is an *arbitrary* $L/2$-periodic function, and $f(t)$ is supersymmetric according to (6). (b) The driving $f(t)$ is time inversion invariant and the potential $V(x, f(t))$ is for every fixed $f(t)$ value supersymmetric in the sense of (4). Note that not only the shape of $V(x, f(t))$ but also the location of the extrema may still be different for any $f(t)$ value. [In other words, a_n and ϕ_n in (5) may now be arbitrary functions of $f(t)$.] The verification that either of these two conditions indeed implies (16) is straightforward. A flurry of interesting and *prima facie* quite unexpected *examples* producing zero current without any fine-tuning of parameters follows immediately.

We close with three remarks: First, the above symmetry and supersymmetry concepts can be readily generalized to higher dimensions and to other classes of overdamped ratchet systems [23]. Examples are dynamics of the form (1) with $f(t) \equiv 0$ but instead with a temporally or spatially periodic modulation of the temperature T in (2). Also the intriguing "accidental" result $\langle \dot{x} \rangle = 0$ for interacting ratchets from [24] can naturally be explained by a generalized supersymmetry argument. Second, besides the remarkable analogy between symmetry and supersymmetry there is also one fundamental difference which appears if an additional inertia term $m\ddot{x}(t)$ is included on the left-hand side of (1) or (15): While symmetry implies $\langle \dot{x} \rangle = 0$ even in the presence of inertia effects, the same conclusion no longer applies in the case of supersymmetry. For *example,* a tilting ratchet (1) with a cosine potential $V(x)$ and a supersymmetric driving $f(t)$ as in Fig. 2b implies $\langle \dot{x} \rangle = 0$ in the overdamped limit but generically $\langle \dot{x} \rangle \neq 0$ if inertia is included. Yet, for any sufficiently small deviations from a perfectly supersymmetric situation, the current $\langle \dot{x} \rangle$ will still be arbitrarily small. In the Hamiltonian limit of vanishing dissipation and thermal noise, a condition reminiscent of supersymmetry has been introduced in [25], while in the intermediate regime of finite inertia and finite dissipation, no comparable symmetry concept is known. Our last remark is that time inversion invariance as well as the condition of detailed balance [16] is not directly related to the symmetry and supersymmetry criteria of our present paper.

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