

Liu replies The preceding Comment [1] contains two main points: (i) The measured force is given by its Eq. (2), a formula derived from free energy balance. (ii) Because there is no unique way to divide Eq. (2) into bulk and surface contributions, information on force densities cannot possibly be retrieved as claimed. The first point is well taken. It is the consideration of [2] that there is a geometry in which the magnetic force is $\int M_k \nabla_i B_k d^3r$, rather than of the Kelvin form, $\int M_k \nabla_i H_k d^3r$, which was verified in [3]. The cited free energy balance shows independently that both the consideration and the experiment are correct. The second point, however, is misguided, because theoretical considerations more detailed than free energy balance exist. These start from the gradient of the stress tensor $\nabla_k \Pi_{ik}$ as the bulk force density, and the stress difference $\Delta \Pi_{nn}$ as the surface force density. [n and t (here and below) denote components normal and tangential to the surface, respectively.]

The canonical view is well rendered in [4], which starts from the magnetic stress, $\Pi_{ik}^{(1)} = H_i B_k - \frac{1}{2} H^2 \delta_{ik}$, and obtains $\nabla_k \Pi_{ik}^{(1)} = M_k \nabla_i H_k$ as the bulk, and $\Delta \Pi_{nn}^{(1)} = \frac{1}{2} M_n^2$ as the surface, force density. Faced with the experiment of [3], because the integrated sum of both force densities matches the result, the obvious conclusion is that the Kelvin form has been verified. This is opposite to our conclusion, but it does show that division and verification are possible (and, of course, so is invalidation).

Our points are: (i) The form of $\pi_{ik}^{(1)}$ is incomplete, and it assumes a special constitutive relation, a slight change of which yields $M_k \nabla_i B_k$ as the bulk force density, while the complete stress renders the total bulk force always zero in equilibrium, leaving only the surface force operative. (ii) This surface force varies with geometry, and may be written as $\int M_k \nabla_i B_k d^3r$ and $\int M_k \nabla_i H_k d^3r$ in two limiting cases. (iii) Generally speaking, the scientific evidence for Kelvin's preference of H over B is flimsy. Because of the symmetric roles that H and B assume in thermodynamics, they also enter the force expressions in a symmetric way. If a special equation of state or geometry leads to the Kelvin form, $M_k \nabla_i H_k$, a very similar one leads to the counterexpression $M_k \nabla_i B_k$ [2,5].

The correct consideration of the magnetic force starts with the total Maxwell stress [2,4],

$$\Pi_{ik}^{\text{tot}} = H_i B_k - \delta_{ik} [P + \frac{1}{2} H^2 + \int dH_i \hat{R} M_i] \quad (1)$$

$$= H_i B_k - \delta_{ik} [P + \frac{1}{2} H^2 - \frac{1}{2} M^2 - \int dB_i \hat{R} M_i], \quad (2)$$

where P is the zero-field pressure, and $\hat{R} \equiv (1 - \rho \partial / \partial \rho)$. Both expressions are algebraically equivalent; the first depends on the temperature T , density ρ , and H_i as independent variables, the second depends on T , ρ , and B_i . The difference in appearance stems from the magnetic contribution to energy, which is $\int H_i dB_i$ and $-\int B_i dH_i$, respectively. Neglecting P and assuming $M \sim \rho$ (i.e., $\hat{R} M = 0$) for given (integration variable) H or B , we either

return to $\Pi_{ik}^{(1)}$ and its above ramifications, especially the Kelvin force, or obtain $\Pi_{ik}^{(2)} = H_i B_k - \frac{1}{2} (H^2 - M^2) \delta_{ik}$, with $\nabla_k \Pi_{ik}^{(2)} = M_k \nabla_i B_k$ as the bulk, and $\Delta \Pi_{nn}^{(2)} = \frac{1}{2} M_t^2$ as the surface, force density. As stated above, different constitutive relations lead to different bulk force densities. (There are signs that the first of the two constitutive relations is better realized in ferrofluids, but this cries out for verification and understanding.)

The crucial point here is that neither was measured in [3] as both expressions neglect the zero-field pressure P . This is a serious omission, because the density is inhomogeneous in the presence of field gradients, such that, ∇P cancels any magnetic bulk force, whatever it may be, and the total volume force vanishes, $\nabla_k \Pi_{ik}^{\text{tot}} = 0$. The pendulum was indeed suspended by a surface force alone.

This surface force is calculated again from the total Maxwell stress and is given as its surface integral $\oint \Delta \Pi_{nn}^{\text{tot}} dA$. It may be written as [2]

$$F = \oint [\int M_n dB_n + \int M_t dH_t] dA, \quad (3)$$

which is the nonlinear version of the starting equation (3) in [3], valid for arbitrary constitutive relations. [The calculation needs the input that $P(T, \rho) - \int (\rho \partial / \partial \rho) M_j dH_j \equiv G(\mu, T)$ is the zero-field Gibbs potential, a function of T and the chemical potential μ , and a spatial constant in equilibrium.] If M is predominantly tangential or normal to the surface, we again have $\oint [\int M dH] dA = \int M \nabla H d^3r$, or $\oint [\int M dB] dA = \int M \nabla B d^3r$, respectively.

It will be of interest to also measure the magnetic bulk force, by tracing the density variation and calculating $\nabla_i P(T, \rho)$ —the quantity compensating the magnetic bulk force. However, note its fictitious character: For constant temperature, we have [2] $\nabla_k \Pi_{ik}^{\text{tot}} = \rho \nabla_i \mu$, with $\nabla_i \mu = (\partial \mu / \partial \rho) \nabla_i \rho + (\partial \mu / \partial H) \nabla_i H$ vanishing in equilibrium. The density inhomogeneity (first term) is a result of the magnetic contribution to the chemical potential (second term). There is no need to take the latter as a force density.

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