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Simultaneous Minimum-Uncertainty Measurement of Discrete-Valued Complementary Observables

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We have made the first experimental demonstration of the simultaneous minimum uncertainty product between two complementary observables for a two-state system (a qubit). A partially entangled two-photon state, where each of the photons carries (partial) information of the initial state, was used to perform such a measurement.

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Complementarity and the associated uncertainty relations play a key role in quantum theory. Uncertainty can be quantified in several ways, e.g., in terms of entropy [1], variance [2–5], or other quadratic functions of measurement probabilities [6,7]. However, what is usually meant by "uncertainty" is the state's *inherent* indeterminism with respect to any two complementary observables [7,8]. A lower bound for this inherent indeterminism is given by the ordinary Schrödinger-Robertson uncertainty relation [2,3], which is based on the variances obtained from a sharp measurement of one of the observables on a particular ensemble, and a sharp measurement of the complementary observable on *another,* but identically prepared ensemble [9].

However, if one wishes to determine the values of two complementary operators on every single member of an ensemble of identically prepared particles, e.g., the value of the spin in both the x and the y directions of a single particle, then additional complications arise. Should one measure the spin x component by making a sharp measurement, then a subsequent measurement of the y component will yield a completely random outcome. Consequently, if we want some correlation between the outcome of the spin y measurement and the preparation of the particle, the measurement of the spin x component must not be sharp.

The first to address this question were Arthurs and Kelly [10,11] who studied canonical observables, such as position and momentum. The objective was to find a minimum uncertainty relation for the observables based on the measurement outcomes of *both* observables on *every* member of the ensemble. However, in a simultaneous measurement one has two conflicting objectives: The measurement variances should be minimized while the correlations between the measurement outcomes and the preparation of the ensemble should be maximized. There is no unique way of doing this. Arthurs and Kelly added an additional requirement, namely, that the expectation values of the two (unsharp) measurements should equal the expectation values of the corresponding *sharp* measurements on any ensemble of identically prepared particles. With this additional requirement Arthurs and Kelly found an uncertainty relation for canonical observables [10] equal in form to, but with a minimum product of the variances 4 times greater than, the Schrödinger-Robertson relation. They also found that in order to achieve this lower bound, *a priori* knowledge about the preparation of the ensemble was needed to adjust the relative sharpnesses of the two measurements. (The sharper one measurement is made, the more unsharp the other must become.) To the best of our knowledge, Arthurs and Kelly's relation has not been tested experimentally.

In an earlier work we have derived a similar simultaneous uncertainty relation for two-state systems, i.e., for two noncanonical observables [5]. Any pure state in the associated Hilbert space can be represented as a point on the Bloch sphere. Suppose we want to measure two complementary observables $\hat{A}$ and $\hat{B}$ and that the Bloch sphere is oriented so that $\hat{A}$'s and $\hat{B}$'s respective eigenstates $|A_+\rangle$, $|A_-\rangle$ and $|B_+\rangle = (|A_+\rangle + |A_-\rangle)/\sqrt{2}$, $|B_-\rangle = (|A_+\rangle - |A_-\rangle)/\sqrt{2}$ lie equidistantly around the equator;

see Fig. 1. Also suppose that the corresponding eigenvalues are $\pm A$ and $\pm B$. It is easy to show that of all the states lying on any specific "longitude" on the Bloch sphere, the state defined by the point where the longitude crosses the equator will have the smallest uncertainty product between $\hat{A}$ and $\hat{B}$. Hence, we restrict our attention to the equatorial states that all have the general form

$$|\psi\rangle = \sqrt{w_{A+}}\,|A_+\rangle \pm \sqrt{w_{A-}}\,|A_-\rangle . \qquad (1)$$

The positive, real numbers w_{A+} and $w_{A-} = 1 - w_{A+}$ are the probabilities of obtaining the results A and $-A$ if we make a sharp $\hat{A}$ measurement on the state.

Hermitian operators with discrete finite spectra cannot be canonical, i.e., have a commutator of the form iC, where C is a real number. Therefore, their minimum uncertainty product is state dependent and not fixed [2–4]. However, for *any* state on the equator (that is, for any value $0 \leq w_{A+} \leq 1$), there exists a combination of simultaneous unsharp $\hat{A}$ and $\hat{B}$ measurements that reaches the minimum variance product dictated by the appropriate simultaneous uncertainty relation [4,5]. It is our objective in this paper to find and make these measurements.

Let us first show that there is no von Neumann measurement that will give us simultaneous knowledge about $\hat{A}$ and $\hat{B}$ and still fulfill our "correct expectation value" requirement. Consider such a measurement of an observable $\hat{D} \neq \hat{A}, \hat{B}$ in the space. The measurement is characterized by an axis through the origin of the Bloch sphere, crossing the sphere shell at points D and $-D$. However, all states in a plane normal to the D axis will give the same measurement probability distributions. In particular, this holds for the two states defined by the crossing points q and $-q$ between the plane and the equator in spite of the fact that they have different values of $\langle\hat{A}\rangle$, $\langle\hat{B}\rangle$, or both. This makes it impossible to fulfill the correct expectation value requirement, so we must rule out any von Neumann measurements and instead look at positive operator valued measurements, POVMs.

To implement a POVM, we need to enlarge our state space. A sufficient way in our case is to introduce an ancillary two-state (probe) particle (the particle whose characteristics we wish to measure is called the object). In order to extract any information about the object from the probe, we must entangle the two particles. We assume that the (unitary) entanglement interaction affects only the probe's state $|m\rangle$, viz.,

$$\begin{aligned} \hat{U}|A_+\rangle \otimes |m\rangle &= |A_+\rangle \otimes |m_+\rangle , \\ \hat{U}|A_-\rangle \otimes |m\rangle &= |A_-\rangle \otimes |m_-\rangle , \end{aligned} \qquad (2)$$

which, for the initial state $|\psi\rangle$, results in the state

$$|\psi_e\rangle = \sqrt{w_{A+}}\,|A_+\rangle \otimes |m_+\rangle \pm \sqrt{w_{A-}}\,|A_-\rangle \otimes |m_-\rangle . \qquad (3)$$

A measure of the entanglement of $|\psi_e\rangle$ is $c = |\langle m_+ \,|\, m_-\rangle|$, where $c = 0$ ($c = 1$) signifies perfect (no) entanglement.

Suppose we decide to infer knowledge about $\hat{B}$ from a direct measurement on the object particle and infer information about $\hat{A}$ (of the object) from the probe particle. Three problems arise: how to ensure that the inferred (and therefore unsharply) measured mean values equal the true (sharp) means, how to chose basis to measure the probe state, and how to optimally choose the entanglement parameter c?

All questions were addressed and answered in [5]. Here we recapitulate only the main conclusions. A sharp measurement of $\hat{B}$ on the object particle in state (3) will yield either of two outcomes B or $-B$ with probabilities w'_{B+} and w'_{B-} that will in general not equal $w_{B+} \equiv |\langle\psi \,|\, B_+\rangle|^2 = 1/2 \pm \sqrt{w_{A+}w_{A-}}$ and $w_{B-} \equiv |\langle\psi \,|\, B_-\rangle|^2 = 1 - w_{B+}$. However, if we associate the measurement outcomes with the rescaled eigenvalues $\pm B/c$, then the expectation values of a sharp rescaled $\hat{B}$ measurement on object particle in state $|\psi_e\rangle$ and a sharp $\hat{B}$ measurement on $|\psi\rangle$ become equal [for any state of the form (3)]. The former measurement is an inferred and unsharp estimation of the latter.

Let us now consider the indirect measurement of $\hat{A}$. Projecting the probe state on either of the projectors $|m_+\rangle\langle m_+|$ or $|m_-\rangle\langle m_-|$, we can get the probabilities w_{A+} or w_{A-}. However, it is not possible to obtain both w_{A+} and w_{A-} exactly by any fixed measurement on the probe state because of the nonorthogonality of $|m_+\rangle$ and $|m_-\rangle$ (if $c \neq 0$). The probabilities will be optimally estimated by using an orthogonal basis derived in [5]. It can be shown that the corresponding von Neumann measurement basis vectors $|M_+\rangle$ and $|M_-\rangle$ then form equal angles γ with the vectors $|m_+\rangle$ and $|m_-\rangle$, respectively (see Fig. 2). The probabilities w'_{A+} and w'_{A-} of obtaining the results M_+ and M_- are not equal to w_{A+} and w_{A-} (unless $c = 0$).

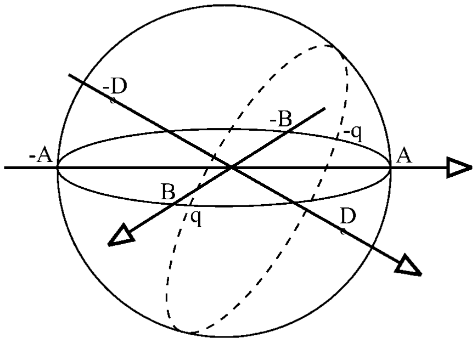

FIG. 1. The Bloch sphere associated with $|\psi\rangle$, $\hat{A}$, and $\hat{B}$.

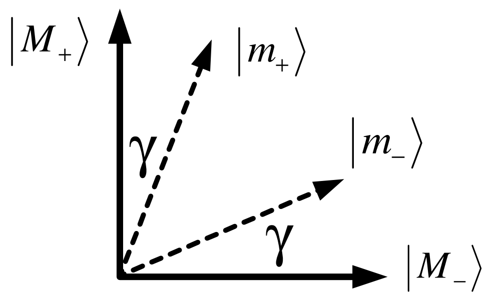

FIG. 2. The probe measurement basis.

However, if we associate the respective probe particle measurement outcomes with the rescaled eigenvalues $\pm A/\sqrt{1 - c^2}$, the inferred (unsharp) measurement of $\hat{A}$ through $|\psi_e\rangle$ will give the true mean of $\hat{A}$ measured on *any* state $|\psi\rangle$ irrespective of the value of w_{A+}.

Now, how do we obtain a minimum uncertainty product for a simultaneous $\hat{A}$ and $\hat{B}$ measurement? For a given state $|\psi\rangle$ the only adjustable parameter in $|\psi_e\rangle$ is the degree of entanglement c. Thus the simultaneous uncertainty product must be minimized with respect to c for a given w_{A+}, i.e., $|\psi\rangle$ must be known *a priori*.

Let $\delta\hat{A} = (\langle\Delta\hat{A}^2\rangle)^{1/2}/A$ and $\delta\hat{B} = (\langle\Delta\hat{B}^2\rangle)^{1/2}/B$ be the normalized uncertainties of $\hat{A}$ and $\hat{B}$ when $|\psi\rangle$ is measured. The normalized uncertainties, inferred from simultaneous measurements on $|\psi_e\rangle$, are denoted $\delta\hat{A}'$ and $\delta\hat{B}'$. A straightforward calculation [5] gives

$$\delta\hat{A}' = \sqrt{(\delta\hat{A})^2 + c^2/(1 - c^2)}, \tag{4}$$

and

$$\delta\hat{B}' = \sqrt{(\delta\hat{B})^2 + (1 - c^2)/c^2}. \tag{5}$$

The product of (4) and (5) reaches the minimum

$$(\delta\hat{A}'\delta\hat{B}')_{\min} = 1 + \delta\hat{A}\delta\hat{B}, \quad \text{for } c = \sqrt{\delta\hat{A}/(\delta\hat{A} + \delta\hat{B})}. \tag{6}$$

These formulas are identical to (53) and (54) in [5] but are expressed in different quantities.

Now we test the simultaneous uncertainty relation with photon states. Unfortunately, state-of-the-art technology does not allow us to perform the unitary interaction (2) on single photons. Therefore, we start directly with state (3) without introducing the independent object and probe states first. This is not as serious a flaw as it may appear. If we had started with $|\psi\rangle$, then we would have had to make sure that the state after the entanglement was indeed $|\psi_e\rangle$, by measuring, e.g., w_{A+}, c and the relative phase angle between the state's two terms (which, if the entangling step works alright, is akin to measuring w_{B+}). Hence, there is actually little point in using a two-step procedure to arrive at $|\psi_e\rangle$ since the parameters w_{A+} and w_{B+} (uniquely defining $|\psi\rangle$) can, and must, be measured from $|\psi_e\rangle$ anyhow.

Experimentally, state (3) was created via noncollinear spontaneous parametric down-conversion with type II phase matching. Our source was similar to that described in [12], but we used a pulsed source to reduce the random coincidence count rate. With a specific choice of the relative phase between the vertically $|\uparrow\rangle$ and horizontally $|\rightarrow\rangle$ polarized photons the state after the crystal is ideally in a mixture of the vacuum state, one photon states (that both are eliminated by postselection through detector correlation), and the state

$$|\psi_i\rangle = (|\uparrow\rangle \otimes |\rightarrow\rangle - |\rightarrow\rangle \otimes |\uparrow\rangle)/\sqrt{2}. \tag{7}$$

If we choose the leftmost ket in the product to represent the object and the rightmost to represent the probe, the state becomes exactly state (3) with $w_{A+} = w_{A-} = \frac{1}{2}$ and $c = 0$. To adjust the state for a minimum simultaneous uncertainty product measurement, a partial polarizer was inserted in the object path, rotated at an angle α with respect to the horizontal plane. The partial polarizer consisted of a stack of N glass plates held at the Brewster angle with respect to the object photon's propagation axis. The amplitude transmittivities of the partial polarizer were $t_\mathrm{p} \approx 1$ and $t_\mathrm{s} = t$, where t was determined by the number of plates N. (The indices p and s refer to the linear polarization states with the respect of the partial polarizer.) Assume that we insert the partial polarizer so that it tends to polarize the object photon vertically (corresponding to $|A_+\rangle$). In this case c will remain zero, but w_{A+} of the *postselected* states will become larger than w_{A-}. If, on the other hand, we make the polarizer perfect and insert it at $\alpha = \pi/4$, then c will become unity but w_{A+} and w_{A-} remain equal. Hence, the degree of entanglement between the object and probe, as well as the *a priori* path probabilities, could be varied by changing N and α. Unfortunately, with this device, it is not possible to vary c and w_{A+} independently. That is, for a given number of plates, only two choices of α give the combination of c and w_{A+} that makes the simultaneous uncertainty product attain its lowest bound.

The experimental setup is shown in Fig. 3. A 0.5 mm long beta-barium borate (BBO) crystal was used to generate noncollinear frequency degenerate photon pairs. The pump was 100 mW average power second harmonic generation from the initial 2 W average power Ti-sapphire pulsed radiation. The experimental parameters were 2 ps pulse duration, 80 MHz repetition rate, and 390 nm wavelength of the second harmonic. The state detection was accomplished by two EG&G single photon detectors (SPC1 and SPC2) with around 60% quantum efficiency and a coincidence counter. The photon pair was selected by two 1 mm diameter irises placed 1 m from the crystal, selecting pairs from a 5° top angle emission cone. Identical 10 nm bandpass filters (F1 and F2) centered at $\lambda = 780$ nm were placed in front of the detectors to select frequency degenerate photon pairs. This resulted in 10 kHz single count events and around 100 Hz of coincidence counts. Two polarizers (P1 and P2) were used to select linearly polarized photons. Polarization rotation of the beams was accomplished by two $\lambda/2$ plates placed before the polarizers. Because the uncertainty product is symmetrical in w_{A+} with respect to the $w_{A+} = 0.5$ point,

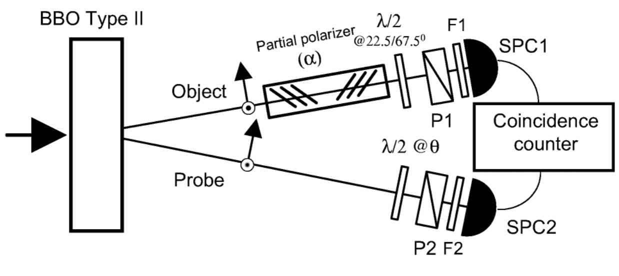

FIG. 3. A schematic illustration of the experimental setup.

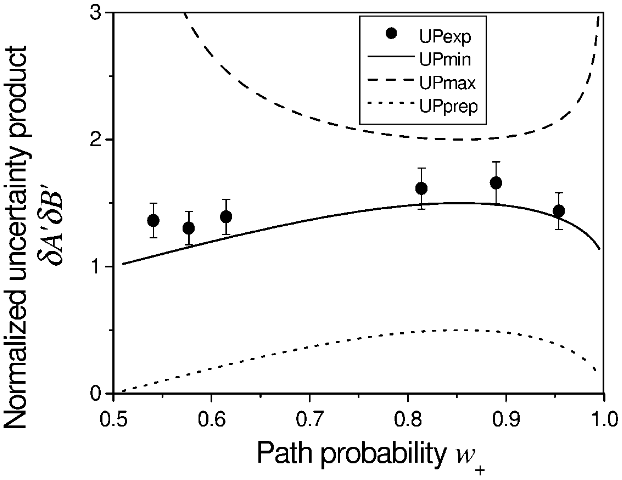

FIG. 4. Results of a simultaneous minimum uncertainty product measurement (circles). The theoretical prediction (6) is shown by the solid line. The maximum uncertainty product (dashed line) and the Schrödinger-Robertson uncertainty relation (dotted line) are given as references.

only the $0.5 \leq w_{A+} \leq 1$ probability range was used for the minimum uncertainty product measurements. Three different numbers of glass plates ($N = 7$, 8, and 10) were used to adjust the state $|\psi_i\rangle$. This choice gave more or less an equally spaced set of probabilities w_{A+}. Thus six combinations of N and α should give the minimum uncertainty product. For a given number of glass plates held at the theoretically computed value of α the four coincidence probabilities with respect to the probe measurement basis orientation were measured, from which the values of c and w_{A+} at this particular setting were derived. This calibration curve allowed us to adjust the partial polarizer rotation angle α and thus the values of c and w_{A+} rather precisely in order to attain the minimum uncertainty product bound.

To measure the $\hat{B}$ outcome probabilities the photon counter, a $\lambda/2$ plate, and a polarizer, were used to project the object state onto the $|B_+\rangle$ and $|B_-\rangle$ eigenstates. (These are the states polarized at 45° and 135° from the horizontal direction.) The $\delta\hat{B}'$ uncertainty was calculated using the measured probabilities with subsequent rescaling due to the measured entanglement parameter c. The $\delta\hat{A}'$ uncertainty was measured by orienting the probe measurement basis to coincide with $|M_+\rangle$ and $|M_-\rangle$ (see Fig. 2) which are also governed by the entanglement parameter c. After subsequent rescaling, the uncertainty product was calculated and plotted in Fig. 4, which is the main result of this paper. The six circles represent the measured uncertainty product. The result is in relatively good agreement with the theory (6) shown by the solid line. One source of errors in the experimentally obtained values originates from the stochastic nature of photocounting events. An estimate of the errors has been done taking into account the errors of the eight measured coincidence coefficient necessary to calculate the uncertainty product. This gives an estimated root mean squared error of 10% of the mean value of the measured uncertainty product. This error is represented by the error bars in Fig. 4.

The degradation of the purity of state (3) results in a systematic error, increasing the uncertainty product. The main reasons for the degradation are the group velocity dispersion in BBO crystals, depolarization effects in our homemade partial polarizer, as well as the light scattering from the numerous optical elements. The degradation affects the $\hat{A}$ measurement to a lesser extent than the $\hat{B}$ measurement as the latter is a relative phase (first order interference) measurement. This results in higher discrepancy between the theory and experiment in the vicinity of the $w_{A+} = 0.5$ point.

The maximum uncertainty product was calculated using the assumption that both the $\hat{A}$ and $\hat{B}$ measurement's two respective outcomes had equal probabilities (that is, as if the initial object state lie at the north or south pole of the Bloch sphere), and using the same rescaling as for the minimum uncertainty product calculation. Then the maximum uncertainty product is given by

$$(\delta\hat{A}'\delta\hat{B}')_{\max} = (c\sqrt{1 - c^2})^{-1} \tag{8}$$

and is shown as a reference by the dashed line.

In conclusion we have experimentally verified, to the best of our knowledge for the first time, a simultaneous uncertainty relation, namely, that between discrete noncanonical observables.

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