

## Does Matter Wave Amplification Work for Fermions?

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We discuss the relationship between bosonic stimulation, density fluctuations, and matter wave gratings. It is shown that enhanced stimulated scattering, matter wave amplification, and atomic four-wave mixing do not require macroscopic occupation of a single quantum state. These processes are in principle possible for fermionic or nondegenerate samples, if they are prepared in a cooperative state. In practice, there are limitations due to short coherence times.

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*Introduction.*—The realization of Bose-Einstein condensation in atoms has made it possible to study the phenomenon of bosonic stimulation for massive particles. Superradiance of atoms [1], four-wave mixing [2], and matter wave amplification [3,4] were described as processes which are bosonically stimulated, i.e., their rates are proportional to  $(N + 1)$ , where  $N$  is the number of identical bosons in the final state. These experimental achievements have raised the question of whether these processes are inherently connected to bosonic systems.

We have recently pointed out that atomic superradiance does not depend on Bose-Einstein statistics and would occur for thermal atoms or even for fermions, although with much shorter coherence times [1], and similar arguments should apply to four-wave mixing. These suggestions have stirred a controversy among researchers. This note will reconcile the different physical descriptions. The central result is that the stimulated processes mentioned above do not rely on quantum statistics, but rather on symmetry and coherence.

In this note, we identify clearly the physical process behind bosonically stimulated scattering. We show that the presence of a macroscopically occupied state increases the density fluctuations of the system, and bosonically enhanced scattering is simply the diffraction of particles from these density fluctuations. The first parts of this paper establish the equivalence of bosonically enhanced scattering, diffraction, and superradiance, which will then be applied to fermionic systems.

*Scattering theory.*—It is useful to summarize basic aspects of the theory of scattering of light or particles from an arbitrary system. These textbook results simply follow from lowest order perturbation theory (Fermi's Golden Rule). The double differential cross section can be decomposed into two factors  $\frac{d^2\sigma}{d\Omega d\omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} S(q, \omega)$ . The first one is the differential cross section for the scattering by a single particle (e.g., the Rayleigh cross section for far-off resonant light scattering), the second one is the dynamic structure factor (van Hove or scattering function)  $S(q, \omega)$  which is the Fourier transform of the density-density correlation function:  $S(q, \omega) =$

$(1/2\pi) \int dt e^{i\omega t} \langle \hat{\rho}(q, t) \hat{\rho}^\dagger(q, 0) \rangle$  where  $\hat{\rho}(q)$  is the Fourier transform of the particle density operator (see, e.g., [5]).

For a noninteracting system of bosons,  $S(q, \omega)$  can be expressed using the single-particle states  $|i\rangle$  with energy  $E_i$  and occupation numbers  $N_i$ :

$$S(q, \omega) = S_0(q)\delta(\omega) + \sum_{i \neq j} |\langle j | e^{iqr} | i \rangle|^2 N_i (N_j + 1) \times \delta[\omega - (E_j - E_i)/\hbar]. \quad (1)$$

The factor  $(N_j + 1)$  reflects bosonic stimulation by the occupation of the final state. The elastic term  $S_0(q)$  describes coherent elastic scattering or diffraction and is simply the square of the Fourier transform of the density  $S_0(q) = |\langle \rho^\dagger(q) \rangle|^2 = |\sum N_i \langle i | e^{iqr} | i \rangle|^2$ .

*A simple example.*—It is instructive to apply this formalism to a system of noninteracting bosons which has macroscopic occupation in two momentum states with momentum  $\pm \hbar k$ . If the initial state is a Fock state  $|+k\rangle^{N_+} |-k\rangle^{N_-}$ , we find that, apart from forward scattering, the dominant term in  $S(q, \omega)$  is the bosonically enhanced scattering between those two (degenerate) states,  $S(q, \omega) = [N^2 \delta_{q,0} + N_+(N_- + 1) \delta_{q,-2k} + N_-(N_+ + 1) \delta_{q,2k}] \delta(\omega)$  where the Kronecker symbol  $\delta_{q,p}$  implies  $q = p$  within the wave vector resolution  $\approx 1/L$  of a finite volume with length  $L$ . Alternatively, we can assume the initial state to be a coherent superposition state  $|i\rangle^N$  with the eigenstate  $|i\rangle = c_+ | +k \rangle + c_- | -k \rangle$  and  $|c_\pm|^2 = N_\pm / N$  and  $N = N_+ + N_-$ . Now, the dominant contribution to  $S(q, \omega)$  comes from  $S_0(q) = N^2 \delta_{q,0} + N^2 |c_+|^2 |c_-|^2 [\delta_{q,2k} + \delta_{q,-2k}]$  which is equivalent to the Fock state case when the difference between  $N_\pm$  and  $N_\pm + 1$  can be neglected in the limit of large occupation numbers.

This equivalence between Fock states and coherent superposition states has been extensively discussed in the context of two interfering Bose-Einstein condensates [6–8] and also with regard to optical coherences [9]. Those papers show that, in many situations, a Fock state is equivalent to an ensemble of coherent states with arbitrary phase.

Experimental interrogation determines the phase and reduces the ensemble to a single coherent state with a phase which will vary from experiment to experiment. For large occupation numbers, one can therefore regard the Fock state as an initial state which has not yet “declared its phase,” and, in many cases, for the convenience of calculations, replace the Fock state by a coherent superposition state with an arbitrarily chosen phase.

However, on first sight, the physical interpretation is different. In the Fock state formulation, the enhanced scattering results from a macroscopic occupation number in a single quantum state, whereas for the coherent superposition state, the scattering is simple diffraction by a sinusoidally modulated density distribution with an amplitude proportional to  $N|c_+c_-|$ . This density modulation acts as a diffraction grating for incident light or particles and has a diffraction efficiency proportional to the square of the amplitude. Such a density modulation does not require bosonic atoms. It can, for example, be imprinted into thermal or fermionic clouds by subjecting them to a suitable optical standing wave. The equivalence of these two descriptions points towards one of the major conclusions of this paper, namely macroscopic population of bosonic states is not necessary for enhanced scattering.

The previous discussion assumed scattering between two degenerate momentum states  $|\pm k\rangle$ . A simple Galilean transformation generalizes this to two arbitrary momentum states  $|k_\pm\rangle$  with energies  $E_\pm$ . Now the standing wave moves with a velocity  $\hbar(k_+ + k_-)/2m$  where  $m$  is the mass of the atoms, and the enhanced scattering appears at  $\hbar\omega = \pm(E_+ - E_-)$  instead of at  $\omega = 0$ .

*Enhancement of fluctuations.*—The general results of statistical physics presented above emphasize that enhanced scattering *must* be related to enhanced density fluctuations. Therefore, bosonic enhancement of a scattering rate is either due to a density modulation  $\langle\rho(q)\rangle$  (in the coherent superposition description) or due to density fluctuations (in the Fock state description)—the latter can be regarded as a density modulation with an unknown phase. This relation allows a more intuitive answer to the question of why there is bosonic enhancement when two atoms 1 and 2 collide in the presence of a condensate with  $N_0$  atoms. The standard answer would be that the symmetry of the wave function enhances the scattering rate into the condensate and into some other state 3 by a factor of  $(N_0 + 1)$ . An equivalent answer is that the condensate interferes with atom 2 (or 1) and creates a density grating with an amplitude proportional to  $N_0^{1/2}$ , which diffracts atom 1 (or 2) into state 3. The grating absorbs this momentum transfer by transferring the atom in state 2 (or 1) into the condensate. Therefore, *bosonic stimulation can be regarded as heterodyne amplification of density fluctuations* where the condensate acts as the local oscillator.

*Dicke superradiance.*—We now want to establish the connection between bosonic enhancement and Dicke superradiance. This will formally introduce the enhance-

ment factor  $(N + 1)$  for nonbosonic systems. A system of  $N$  atoms in two states  $|\pm\rangle$  is conveniently described with the formalism introduced by Dicke to discuss superradiance in two-level atoms [10]. It should be emphasized that the only assumption in this treatment is that the  $N$  atoms couple identically to the probe field (the electromagnetic field or some incident particle beam), i.e., that they have the same transition frequency and matrix element without any assumption of quantum statistics. For example, in magnetic resonance experiments, the Dicke treatment would apply to different atomic species with the same value of the magnetic moment.

Dicke regarded the two-level atom as a spin  $1/2$  system and introduced angular momentum quantum numbers. In this subspace, a fully symmetric state of  $N$  atoms has spin  $s = N/2$  and magnetic quantum number  $m = (N_+ - N_-)/2$ . The squared matrix element for the transition  $|s, m \pm 1\rangle \rightarrow |s, m\rangle$  induced by the ladder operator  $S_\mp$  is  $(s \pm m + 1)(s \mp m)$ . Expressing this by initial occupation numbers  $N_\pm$ , one obtains  $N_\pm(N_\mp + 1)$  [11–13], retrieving the formula of bosonic enhancement. For a given  $N$  and  $m$ , the transition rates are largest for the state with  $s = N/2$  which is therefore called the state of maximum cooperativity.

Such a system will couple to the probe field in a superradiant way (i.e., with an up to  $N$  times enhanced transition rate). In the Bloch vector picture, its dynamics is described as the precession of a macroscopic spin vector with length  $s = N/2$ . This spin vector decays in a time  $1/\Gamma$  where  $\Gamma$  is the total (homogeneous and inhomogeneous) linewidth of the transition  $|+\rangle \rightarrow |-\rangle$ . Collective superradiant behavior can be observed only at times shorter than  $1/\Gamma$ .

*Matter wave gratings and fermions.*—Dicke’s formalism is usually applied to one-photon transitions between internal states, but here we use it to discuss scattering, i.e., a two-photon transition between two momentum states  $|k_\pm\rangle$ . Let us first assume that we have an ideal Bose-Einstein condensate in the  $k = 0$  momentum state. Light scattering between momentum states  $k = 0$  and  $k = q$  has an infinite coherence time for a noninteracting condensate of infinite size (Fig. 1a). For a thermal (nondegenerate) cloud of atoms with thermal momentum spread  $\hbar k_{\text{th}} \ll \hbar q$  the transition frequencies for the transfer of momentum  $\hbar q$  are Doppler broadened by  $\Gamma = \hbar k_{\text{th}} q/m$ . For times shorter than  $1/\Gamma$  the system will behave collectively like the Bose condensed system, i.e., a probe beam would induce transitions between the  $k = 0$  and  $k = q$  momentum states at a rate proportional to  $N_{k=0}(N_{k=q} + 1)$  where  $N_{k=0,q}$  refers to the total number of atoms in states with momentum around  $k = 0, q$ .

Once we have distributed the particles over many initial states, indistinguishability and quantum statistics do not play any role. Therefore, the only modification for a Fermi degenerate cloud is to replace  $k_{\text{th}}$  with the Fermi wave vector  $k_F$  in the expression for the inhomogeneous broadening (Fig. 1b). Because of the assumption  $\hbar k_F \ll \hbar q$ , Pauli

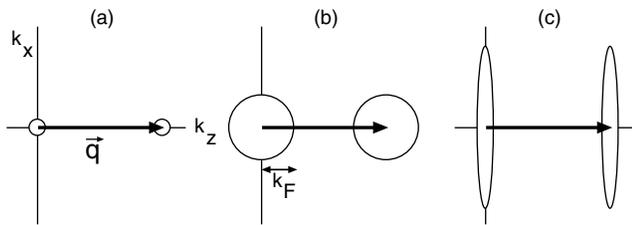


FIG. 1. Momentum transfer  $\vec{q}$  (a) to a Bose-Einstein condensate, (b) to a Fermi sea, and (c) to a momentum squeezed degenerate Fermi cloud. Shown are the populated states vs the  $k$  vector. The momentum spread  $k_F$  of the Fermi sea introduces Doppler broadening of the transition and a finite coherence time, whereas the coherence times in (a) and (c) are much longer due to small momentum spreads in the  $z$  direction and could in principle be infinite.

blocking due to scattering into already occupied states is absent. If this assumption is not made, a part of the cloud becomes inactive, and our discussion would apply only to the atoms near the Fermi surface.

The previous paragraph generalized the bosonic *Fock state* ensemble to nondegenerate and fermionic clouds. We now come back to the *coherent superposition* state. For bosons, it can be produced from a Bose-Einstein condensate in the  $\vec{k} = \vec{0}$  state by applying a (so-called Bragg) pulse of two laser beams which differ in wave vector by  $\vec{q}$  and in frequency by the recoil frequency  $\hbar q^2/2m$ . Those beams resonantly drive the transition between momentum states  $\vec{k} = \vec{0}$  and  $\vec{k} = \vec{q}$  [14,15] and prepare the superposition state discussed above. Similarly, in a thermal (or fermionic) cloud, the Bragg pulse creates a modulated density distribution with wavelength  $2\pi/q$  which has the same contrast as in the bosonic case and will diffract light or atoms at the same rate. However, due to the thermal motion with velocity  $\hbar k_{th}/m$ , this grating decays during a time  $m/\hbar k_{th}q = 1/\Gamma$  (for the fermionic case,  $k_F$  has to be substituted for  $k_{th}$ ). Thus the Dicke picture and the diffraction picture agree.

*Coherence time.*—The Doppler broadening discussed above seems to imply a fundamental limit to the coherence time of a Fermi system. However, at least in principle, one can prepare a Fermi system with infinite coherence time by starting out with a cloud which is in a single momentum state along the  $\hat{z}$  axis, but occupies many momentum states along  $\hat{x}$  and  $\hat{y}$ . With a Bragg pulse transferring momentum  $q\hat{z}$ , one can prepare a system which shows collective behavior for scattering particles or light with momentum transfer  $q\hat{z}$  with an infinite coherence time (Fig. 1c). Therefore, there is no direct connection between a long coherence time and a high phase-space density. In this ensemble, the scattering is between the states  $|k_z = 0\rangle \otimes |k_x, k_y\rangle$  and  $|k_z = q\rangle \otimes |k_x, k_y\rangle$ . Therefore, we have enhanced scattering into the  $|k_z = q\rangle$  quantum state, but the atoms may differ in other quantum numbers. What matters is only the symmetrization of the many-body wave function along  $\hat{z}$ . The other quantum

numbers ensure that there is no conflict with the Pauli blocking for fermionic systems. This is analogous to the separation of electronic wave functions into a symmetric part (e.g., the spin part) and an antisymmetric part (e.g. the spatial part) where the coupling to an external field (e.g., electron spin resonance experiment) depends only on the symmetric part.

*Experiments.*—The experiments both on superradiance [1] and four-wave mixing [2] in Bose-Einstein condensates have in common that a matter wave grating formed by two macroscopically occupied momentum states is probed, either by light or by atoms. Both experiments create the coherent superposition state discussed above using a Bragg pulse. In the limit of low intensity of the probe beam, the scattering is independent of the nature of the probe particles—one could have used any kind of radiation, bosons or fermions [16]. The bosonic stimulation observed in both experiments demonstrates the dynamic nature of the matter wave grating. Each time a particle or photon is diffracted, the amplitude of the grating grows.

In practice, it is difficult or impossible to carry out these experiments with fermions or thermal atoms. When we observed superradiance of a condensate, we could not observe similar behavior above the BEC transition temperature since the threshold laser intensity for superradiant gain is several orders of magnitude higher (see Ref. [1] for details). Furthermore, the superradiance may be suppressed by heating or other decoherence processes. The shorter coherence time for non-BEC samples should be even more crucial for the four-wave mixing experiment where the matter wave grating is probed by very slow atoms which have a long transit time of about 1 ms through the sample. Another concern is incoherent scattering of the probe particles which accompany the stimulated processes discussed so far. Since the incoherent processes scale linearly with the number of atoms, whereas the stimulated process is proportional to  $N^2$ , there is in principle always a regime where the stimulated process dominates [17].

*Discussion.*—Coming back to the initial question: Is matter wave amplification possible for fermions? The answer is yes, if the system is prepared in a cooperative state and the amplification occurs in a time shorter than the coherence time. However, this amplification does not pile up atoms in a single quantum state, but rather in states which are in the same (or approximately the same) momentum state along  $\hat{z}$ , but differ in other quantum numbers. Therefore, this amplification can be regarded as amplification of a density modulation or as amplification of spatial bunching of atoms. Alternatively, one can regard the density modulation as a collective excitation of the system which involves bosonic quasiparticles (e.g., phonons). Superradiance and four-wave mixing (both with bosons and fermions) can then be ascribed to bosonic stimulation by those quasiparticles.

The phase-coherent matter wave amplification for fermions would start with a short Bragg pulse which puts

some of the atoms into a recoil state which is then amplified. This superposition of two momentum states creates a matter wave grating. This can be regarded as the interference pattern of each atom with itself with all the individual interference patterns being exactly in phase. Matter wave amplification occurs when a single laser beam is diffracted off this grating, increasing the amplitude of each atom to be in the recoiling state. Therefore, the matter wave amplification scheme of Refs. [3,4] would work for fermions, provided the whole process can be done in the short coherence time of the fermionic matter wave grating.

Of course, there is a fundamental difference between bosons and fermions which is reflected in the symmetry of the total wave function. A bosonic system with two macroscopically occupied quantum states is *always* in a fully symmetric and maximally cooperative state. In other words, if two independent Bose condensates cross each other, there is always a macroscopic interference pattern (as observed experimentally [18]), which is reflected in  $S(q, \omega)$  being proportional to  $N^2$  (or to  $N_+N_-$ , to be more precise). It is this density modulation which can be amplified by the dynamic diffraction discussed in this paper. If two beams of fermions overlap, there is no macroscopic interference, unless the two beams were prepared in a symmetric way, e.g., by generating one of the beams by a Bragg pulse from the other one.

Our discussion of scattering without change of the internal state can be generalized. For example, if atoms scatter into the condensate through a spinflip process, the density grating has to be replaced by a polarization or coherence grating. Such gratings were experimentally studied for laser-cooled atoms in Ref. [19].

This paper has focused on bosonically enhanced *scattering*. Similarly, bosonic enhancement of spontaneous emission can be equally well described by a cooperative initial state without invoking quantum statistics. For scattering, the relevant coupling strength is density fluctuation. For spontaneous emission, it is the electric dipole moment. Both are enhanced by the presence of a Bose condensate, in the latter case because the excited atom corresponds to a Dicke vector of spin  $s = N/2, m = -N/2 + 1/2$  which couples more strongly to the vacuum fluctuations of the electromagnetic field than an individual atom.

In conclusion, we have shown that bosonically enhanced scattering is related to density fluctuations and matter wave gratings. The analogy with Dicke superradiance emphasizes that matter wave amplification and atomic

four-wave mixing are possible for fermionic or nondegenerate samples if they are prepared in a cooperative state which shows coherent and collective behavior.

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*Note added.*—After submission of this paper we learned of similar work by Moore and Meystre [20] which agrees with our conclusions.

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