Atomic Four-Wave Mixing: Fermions versus Bosons

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We compare four-wave mixing in quantum degenerate gases of bosonic and fermionic atoms. We find that matter-wave gratings formed from either bosonic or fermionic atoms can in principle exhibit nearly identical Bragg scattering and four-wave mixing properties. This implies that effects such as coherent matter-wave amplification and superradiance can occur in degenerate Fermi gases. This effect is due to constructive many-particle quantum interferences, which in the boson case are interpreted as "Bose enhancement."

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The experimental realization of *nonlinear atom optics* [1,2] is one of many recent advances made possible by the achievement of atomic Bose-Einstein condensation (BEC). The demonstration of atomic four-wave mixing [3], the discovery of BEC superradiance [4,5], and the development of matter-wave amplification [6,7] are all examples of nonlinear wave mixing involving atomic Bose-Einstein condensates. Following these remarkable BEC experiments, questions concerning the role of "bosonic stimulation" and the possibility to observe four-wave mixing with fermionic atoms have been debated throughout the BEC community.

A prototypical four-wave mixing experiment involves two basic elements: a two-body ("nonlinear") interaction, and three initial input "waves" (waves 1–3). Four-wave mixing can be said to occur if the amplitude of a distinct fourth wave (wave 4), whose properties can be predicted knowing those of the three initial waves, is much larger than the background amplitude corresponding to particles scattered into other accessible final states. For example, in matter-wave amplification [6,7] one optical wave and two matter waves are input, resulting in the generation of a new optical wave and the amplification of one matter wave at the expense of the other. Atomic four-wave mixing [3] works similarly, but with four atomic matter waves. In BEC superradiance [4], on the other hand, only two waves are input, one atomic and one optical, and both a new atomic and a new optical field are generated via a similar four-wave mixing process.

The four-wave mixing process begins when a particle from wave 1 "collides" with a particle from wave 2. If one of these particles is scattered into wave 3, momentum conservation guarantees that the other is scattered into 4. Energy and momentum conservation, however, also permit both particles to scatter into final states not associated with waves 3 or 4. These two outcomes are distinguished only by the fact that wave 3 has nonzero amplitude to begin with. Under suitable circumstances, the first outcome is seen to dominate the latter, an effect which in the BEC regime is readily interpreted as being due to bosonic stimulation. A complementary interpretation is that waves 2 and 3 interfere and form a matter-wave grating that scatters a particle from wave 1 into 4. The first mechanism would seem to require a BEC, whereas the second should occur even in an ultracold gas of fermions. In this Letter we reconcile these two viewpoints by considering whether or not significant qualitative differences would have been observed had the experiments [3,4,6,7] been conducted instead with a gas of fermionic atoms, which we take for simplicity to be at temperature $T = 0$. The generalization of our results to finite temperature Bose and Fermi systems is straightforward.

Our analysis shows that the establishment of a high-quality fermionic grating is clearly possible, and furthermore leads to scattering properties practically indistinguishable from those of a BEC with the same mean density profile. That is, the four-wave mixing efficiency is the same in both cases. We then present a quantum-mechanical interpretation of this result, contrasting the Bose stimulation responsible for four-wave mixing in the boson case to a collective quantum interference effect, closely related to Dicke superradiance [8], in the fermion case. Finally, we propose a new set of experiments in which four-wave mixing will be observed in the case of a BEC, yet analogous phenomena will not occur for fermions.

We recall that Bose stimulation, a quantum statistical effect that occurs when many bosons occupy a single quantum state, causes the transition amplitude for the process $\hat{\mathcal{V}}|1,N\rangle \rightarrow |0,N+1\rangle$ to be proportional to $\sqrt{N+1}$. Collective quantum interference, on the other hand, occurs when a many-particle system is prepared in a quantum superposition of states which are each dynamically transformed into the same final state by some interaction. The transition amplitude is then the sum of the amplitudes for each "path" and (after accounting for normalization factors) is proportional the square root of the number of distinct initial states. These two collective effects are, however, closely related: when viewed in "first-quantized" form, the Bose-stimulated process is also revealed as a many-particle quantum interference effect, where many initial states (the $N + 1$ different terms under exchange of particle labels) lead to a single final state. Thus while four-wave mixing phenomena with fermionic matter waves may be viewed as collective effects rather than as stimulated processes, at the most fundamental level such a distinction is not necessarily meaningful.

In order to demonstrate that "four-wave mixing" effects analogous to those seen in Bose condensed systems occur as well with fermionic matter waves, and in particular to show exactly how collective (superradiant) states are created, we now examine in detail a simple model system of interacting atomic matter waves. This system consists of two scalar fields with annihilation operators denoted as $\hat{\Psi}_1(\mathbf{r})$ and $\hat{\Psi}_2(\mathbf{r})$. The first field contains N identical bosonic or fermionic atoms from which a matter-wave grating is formed. The second field contains a single test particle which will probe the scattering properties of the grating. This test particle might be an atom, in which case the scattering properties are related to atomic four-wave mixing experiments, or it could be a photon, in which case the results would relate to phenomena such as BEC superradiance and matter-wave amplification. The two fields are subject to the free Hamiltonians $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$, respectively, and are coupled via a two-body interaction of the form

$$\hat{\mathcal{V}} = \lambda \int d^3r \hat{\Psi}_1^\dagger(\mathbf{r}) \hat{\Psi}_2^\dagger(\mathbf{r}) \hat{\Psi}_2(\mathbf{r}) \hat{\Psi}_1(\mathbf{r}), \qquad (1)$$

which describes equally well atom-atom collisions in the s-wave scattering approximation or the effective interaction between ground-state atoms and far off-resonant photons. For simplicity we neglect collisions between atoms in the grating.

By definition, the matter-wave grating consists of atoms in two distinct momentum groups, the wave vector of the grating being their separation in $\mathbf{k}$ space. One can consider these two momentum groups as the two input "beams" for four-wave mixing. The third input "wave" is the initial state of the test particle and the output (fourth) wave is this state shifted in momentum by the wave vector of the grating. Four-wave mixing occurs if the scattering cross section of the test particle is predominately into this final state, i.e., first order Bragg scattering from the matter-wave grating dominates background scattering.

In each of the aforementioned experiments the input matter waves were formed from a single initial BEC by use of a "beam splitter" based on laser-driven two-photon Bragg transitions between center-of-mass states [9]. Each atom was thereby transferred into a coherent superposition of its initial state and a copy of that state displaced in momentum space by the two-photon recoil kick. Single-particle quantum interference between these two momentum groups then results in a typical "standing wave" density modulation. We note that this coherent beam-splitting technique is critical to our claim that similar experimental results could have been obtained with fermions, as this prepares the necessary collective state.

For the purpose of this Letter, we specialize to the case of gratings formed from harmonically confined gases. We first introduce the three-dimensional harmonic oscillator states

$$\varphi_m(\mathbf{r}) = \phi_{\alpha(m)}(x/\xi_x)\phi_{\beta(m)}(y/\xi_y)\phi_{\gamma(m)}(z/\xi_z), \qquad (2)$$

where ξ_j is the oscillator length along the j axis, $\alpha(m)$, $\beta(m)$, and $\gamma(m)$ are the quantum numbers of the mth 3D oscillator energy level, and ϕ_n is the normalized nth 1D harmonic oscillator energy level. We further define the creation operators $\hat{a}_m^\dagger(\mathbf{k})$ for the *momentum side modes* of the mth oscillator state as

$$\hat{a}_m^\dagger(\mathbf{k}) = \int d^{3r} \varphi_m(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\Psi}_1^\dagger(\mathbf{r}). \qquad (3)$$

At $T = 0$ the states of N-atom Bose and Fermi gases are $[\hat{a}_0^\dagger(0)]^N|0\rangle/\sqrt{N!}$ and $\prod_{m=0}^{N-1} \hat{a}_m^\dagger(0)\,|0\rangle$, respectively. Thus immediately after a "50/50" Bragg pulse is applied, the state of the matter-wave grating is given for bosons by

$$|\psi_B\rangle = [2^N N!]^{-1/2}[\hat{a}_0^\dagger(0) + e^{i\theta}\hat{a}_0^\dagger(\mathbf{K})]^N|0\rangle, \qquad (4)$$

and for fermions by

$$|\psi_F\rangle = [2^N]^{-1/2} \prod_{m=0}^{N-1} [\hat{a}_m^\dagger(0) + e^{i\theta}\hat{a}_m^\dagger(\mathbf{K})]\,|0\rangle, \qquad (5)$$

where θ and $\hbar\mathbf{K}$ are the relative phase and momentum transfer imparted on the atoms by the Bragg splitter. Henceforth we assume $\theta = 0$ for simplicity. We note that these states are normalized only in the limit that $\hbar\mathbf{K}$ is much larger than the rms spread in momentum of the initial Bose and Fermi clouds, a condition which we assume to hold. It is this condition, in fact, which is sufficient to guarantee a high-quality grating at wavelength $2\pi/K$.

To compare the quality of the gratings we compute the mean atomic densities $\rho(\mathbf{r}) = \langle \hat{\Psi}_1^\dagger(\mathbf{r})\hat{\Psi}_1(\mathbf{r})\rangle$ for the two states (4) and (5). By making use of the well-known (anti)-commutation relations for bosons and fermions we find

$$\rho_B(\mathbf{r}) = N|\varphi_0(\mathbf{r})|^2[1 + \cos(\mathbf{K}\cdot\mathbf{r})] \qquad (6)$$

and

$$\rho_F(\mathbf{r}) = \sum_{m=0}^{N-1} |\varphi_m(\mathbf{r})|^2[1 + \cos(\mathbf{K}\cdot\mathbf{r})]. \qquad (7)$$

This shows that in both cases, the effect of the Bragg splitter is to superpose a density modulation $[1 + \cos(\mathbf{K}\cdot\mathbf{r})]$ to the mean density of the initial atomic cloud. The modulation depth is unity because we have chosen arbitrarily to split each atom into an equal superposition of the two momentum groups. By choosing different harmonic trap widths for the two systems it is possible to create atomic clouds with both the same atom number as well as the same rms widths in $\mathbf{r}$ space, in which case the mean density profiles would appear nearly identical. Because the spatial densities are then approximately the same, the higher phase-space density of the BEC implies that the boson momentum groups are significantly more localized in momentum space. Doppler broadening has therefore a stronger negative effect on the lifetime of the fermion grating, potentially making the experimental observation of four-wave mixing more difficult.

The next step is to compare the scattering properties of these two gratings. We consider the case where the single test particle is incident on the grating with wave vector $\mathbf{k}_0$ and use perturbation theory to compute the probability $P(\mathbf{k}, t)$ that it is scattered into a given state $\mathbf{k}$ after some time t. The initial state of the system is

$$|\psi(0)\rangle = \hat{c}^\dagger(\mathbf{k}_0)\,|\psi_{B,F}\rangle, \tag{8}$$

where

$$\hat{c}^\dagger(\mathbf{k}_0) = V^{-1/2}\int d^{3r}\, e_0^{i\mathbf{k}\cdot\mathbf{r}}\hat{\psi}_2^\dagger(\mathbf{r}) \tag{9}$$

creates a plane-wave test particle of momentum $\mathbf{k}_0$ in the quantization volume V, and clearly,

$$P(\mathbf{k}, t) = \langle\psi(t)|\hat{c}^\dagger(\mathbf{k})\hat{c}(\mathbf{k})\,|\psi(t)\rangle, \tag{10}$$

where $|\psi(t)\rangle$ is the solution of the Schrödinger equation. This scattering problem can be solved perturbatively by expanding $|\psi(t)\rangle$ to first order in the parameter λ of Eq. (1). This yields the scattering probabilities after the test particle has been scattered once by the grating.

In order to proceed we must now specify the free Hamiltonians $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$. Our approach is to assume that the states $\varphi_m(\mathbf{r})\exp[i\mathbf{k}\cdot\mathbf{r}]$ are approximate eigenstates of the first-quantized version of $\hat{\mathcal{H}}_1$, with energy $E_m(\mathbf{k}) \approx= \hbar\omega_m + \hbar\omega_1(\mathbf{k})$ where $\hbar\omega_m$ is the energy of the mth trap eigenstate and $\omega_1(\mathbf{k}) = \hbar k^2/2M$, M being the atomic mass. These states are essentially low lying levels of the harmonic trap shifted in momentum space by $\hbar\mathbf{k}$. For ultracold atoms it is reasonable to neglect the evolution of these wave packets for times small compared to the oscillator length divided by the velocity $\hbar k/m$. For these short times the overlap between the initial wave packet and the state it evolves into remains approximately unity. In addition, we take the plane waves $(1/\sqrt{V})\exp[i\mathbf{k}\cdot\mathbf{r}]$ to be eigenstates of the first-quantized version of $\hat{\mathcal{H}}_2$ with energy $\omega_2(k)$.

For $\mathbf{k} \neq \mathbf{k}_0$ and $|\mathbf{K} - \mathbf{k}_0| \sim K$ we find to leading order in λ

$$P_{B,F}(\mathbf{k}, t) \approx \frac{\lambda^2}{2\hbar^2V^2}\big[|F(\mathbf{k}, 0, t)|^2N + |F(\mathbf{k}, \mathbf{K}, t)|^2N + |F(\mathbf{k}, 0, t) + F(\mathbf{k}, \mathbf{K}, t)|^2\mathcal{G}_{B,F}(\mathbf{k} - \mathbf{k}_0) + |F(\mathbf{k}, \mathbf{K}, t)|^2\mathcal{G}_{B,F}(\mathbf{k} - \mathbf{k}_0 - \mathbf{K})\big]. \tag{11}$$

Here, the time-dependent function

$$F(\mathbf{k}, \mathbf{k}', t) = \frac{\sin[\Omega_-(\mathbf{k}, \mathbf{k}')t/2]}{\Omega_-(\mathbf{k}, \mathbf{k}')t/2}\, e^{-i\Omega_+(\mathbf{k}, \mathbf{k}')t} \tag{12}$$

gives the effects of energy conservation, where $\Omega_\pm(\mathbf{k}, \mathbf{k}') = \omega_1(\mathbf{k}' + \mathbf{k}_0 - \mathbf{k}) + \omega_2(\mathbf{k}) \pm \omega_1(\mathbf{k}') \pm \omega_2(\mathbf{k}_0)$, while the functions

$$\mathcal{G}_B(\mathbf{q}) = \frac{1}{2}N(N - 1)\,|G_{00}(\mathbf{q})|^2, \tag{13}$$

and

$$\mathcal{G}_F(\mathbf{q}) = \frac{1}{2}\left[\left|\sum_{m=0}^{N-1}G_{mm}(\mathbf{q})\right|^2 - \sum_{m,n=0}^{N-1}\left|G_{mn}(\mathbf{q})\right|^2\right], \tag{14}$$

where

$$G_{mn}(\mathbf{q}) = \int d^{3r}\,\varphi_m^*(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\varphi_n(\mathbf{r}), \tag{15}$$

describe the shapes of the Bragg resonances. The approximate equality in Eq. (11) indicates that we have dropped several (negligible) terms from the exact expression.

The first two terms in Eqs. (11) correspond to spontaneous scattering, whereas the third and fourth terms correspond to 0th order (small angle) and 1st order Bragg resonances, respectively. We note that $\mathbf{k}_0$ must be chosen such that $F(\mathbf{k}, \mathbf{K}, t)$ overlaps with the first-order Bragg resonance, i.e., the usual Bragg condition on the incident wave vector is assumed. The first important feature of $P(\mathbf{k}, t)$ is the scaling of the peak of the Bragg resonance. By setting $\mathbf{q} = 0$ in Eqs. (13) and (14) we see that the Bragg maxima scale as N^2, whereas the background scattering scales as N. In the BEC case, this scaling is readily understood: in order to Bragg scatter the test particle, an atom from the momentum group at $\mathbf{k} = \mathbf{K}$ must be transferred to the momentum group at $\mathbf{k} = 0$ to conserve momentum. Hence the enhancement factor of N over the background is due to bosonic stimulation, the final state being already occupied on average by $N/2$ atoms. For the fermionic case the N^2 scaling can be due to only collective effects, e.g., superradiant spontaneous emission also scales as N^2. We state without proof that despite the more complicated mathematical form of Eq. (14), not only the Bragg maximum, but also the entire shape of the resonance is nearly identical for both $\mathcal{G}_B(\mathbf{k})$ and $\mathcal{G}_F(\mathbf{k})$, provided only that the rms widths and the mean densities (6) and (7) are the same. This has been verified by plotting Eq. (11) for $N = 86$ atoms in spherically symmetric traps with different oscillator lengths chosen so that the clouds have the same spatial extent. Thus we have the perhaps surprising result that the two microscopically very different gratings produce almost identical scattering cross sections. We remark that for $N = 1$ there is no Bragg resonance, only spontaneous scattering. This shows that the Bragg resonance is a *many-particle* effect; a single atom whose wave function is periodically modulated will not Bragg scatter, and hence cannot be considered as a grating.

While both types of gratings result in practically identical scattering distributions, the underlying physical processes seem quite different at first. We can gain considerable physical intuition about them by considering the simple case $N = 2$ in more detail. In this case, the initial states of the Bose and Fermi gratings are found by expanding Eqs. (4) and (5), yielding

$$|\psi_B\rangle = \frac{1}{\sqrt{8}}[\hat{a}_0^{\dagger 2}(\mathbf{0}) + 2\hat{a}_0^\dagger(0)\hat{a}_0^\dagger(\mathbf{K}) + \hat{a}_0^{\dagger 2}(\mathbf{K})]|0\rangle \tag{16}$$

and

$$|\psi_F\rangle = \frac{1}{2}[\hat{a}_0^\dagger(0)\hat{a}_1^\dagger(0) + \hat{a}_0^\dagger(0)\hat{a}_1^\dagger(\mathbf{K}) + \hat{a}_0^\dagger(\mathbf{K})\hat{a}_1^\dagger(0) + \hat{a}_0^\dagger(\mathbf{K})\hat{a}_1^\dagger(\mathbf{K})]|0\rangle. \tag{17}$$

In this representation the gratings appear as quantum superpositions of different initial states. As a result of energy conservation, Bragg scattering occurs only when one of the atoms in group $\mathbf{K}$ scatters the test particle. Scattering from this group is described by acting on the initial state with the operator $\sum_m \hat{a}_m^\dagger(\mathbf{k})\hat{a}_m(\mathbf{K})$, resulting for bosons in the state

$$|\phi_B\rangle = \frac{1}{\sqrt{2}}[\hat{a}_0^\dagger(0)\hat{a}^\dagger(\mathbf{k}) + \hat{a}_0^\dagger(\mathbf{k})\hat{a}_0^\dagger(\mathbf{K})]|0\rangle, \tag{18}$$

and for fermions

$$|\phi_F\rangle = \frac{1}{2}[\hat{a}_0^\dagger(0)\hat{a}_1^\dagger(\mathbf{k}) + \hat{a}_0^\dagger(\mathbf{k})\hat{a}_1^\dagger(0) + \hat{a}_0^\dagger(\mathbf{k})\hat{a}_1^\dagger(\mathbf{K}) + \hat{a}_0^\dagger(\mathbf{K})\hat{a}_1^\dagger(0)]|0\rangle. \tag{19}$$

The normalized final state of the system is therefore $|\psi'_{B,F}\rangle = |\phi_{B,F}\rangle/\sqrt{\langle\phi_{B,F}|\phi_{B,F}\rangle}$.

The matrix element for this scattering process is given by

$$\left\langle \psi'_{B,F} \left| \sum_m \hat{a}_m^\dagger(\mathbf{k})\hat{a}_m(\mathbf{K}) \right| \psi_{B,F} \right\rangle = \sqrt{\langle\phi_{B,F}|\phi_{B,F}\rangle}. \tag{20}$$

Far from the Bragg resonance we can safely assume $[\hat{a}_m(\mathbf{k}), \hat{a}_n^\dagger(0)]_\pm \approx 0$ and $[\hat{a}_m(\mathbf{k}), \hat{a}_n^\dagger(K)]_\pm \approx 0$, in which case the matrix elements are unity for both the boson and fermion gratings. At the Bragg resonance ($\mathbf{k} = 0$), however, the first term in Eq. (18) becomes $\hat{a}_0^{\dagger 2}(0)|0\rangle$, i.e., a particle is transferred into an already occupied state. Bose enhancement then results in a matrix element of $\sqrt{3/2}$. For fermions, the first two terms in Eq. (19) become identical and thus their amplitudes interfere constructively, also resulting in a matrix element of $\sqrt{3/2}$. Thus the quantum state of the grating (17) corresponds to a Dicke collective state symmetric with respect to which harmonic oscillator state is in which momentum group. We emphasize that the state of the grating is only a Dicke "superradiant" state with respect to scattering in one direction; there are many scattering directions where it is not a superradiant state. This additional feature is analogous to superradiant spontaneous emission from atoms with multiple ground states.

We began by pointing out that Bose stimulation occurs because quantum degenerate bosonic states are analogous to Dicke states. The unique thing about bosons is that this enhancement is guaranteed for transitions between macroscopically occupied quantum states, whether one population is coherently split off of the other or formed independently in another apparatus. This is not true for fermions, where the observation of four-wave mixing requires the special preparation of a collective state. A second class of experiments, where the various matter waves consist of independently generated degenerate atomic ensembles, should therefore show a distinct contrast between bosons and fermions. While four-wave mixing will still occur in the BEC case, it will no longer be observed with fermions. Instead a hole, or anti-four-wave mixing, might be observed in the background scattering probability as Fermi blocking prevents Bragg scattering.

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