Entangled Quantum States as Direction Indicators

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We consider the use of N spin-$\frac{1}{2}$ particles for indicating a direction in space. If $N > 2$, their optimal state is entangled. For large N, the mean square error decreases as N^{-2} (rather than N^{-1} for parallel spins).

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Information theory usually deals with the transmission of a sequence of discrete symbols, such as 0 and 1. Even if the information to be transmitted is of a continuous nature, such as the position of a particle, it can be represented with arbitrary accuracy by a string of bits. However, there are situations where information cannot be encoded in such a way. For example, the emitter (conventionally called Alice) wants to indicate to the receiver (Bob) a direction in space. If they have a common coordinate system to which they can refer, or if they can create one by observing distant fixed stars, Alice simply communicates to Bob the components of a unit vector $\boldsymbol{n}$ along that direction, or its spherical coordinates θ and ϕ. But if no common coordinate system has been established, all she can do is to send a real physical object, such as a gyroscope, whose orientation is deemed stable.

In the quantum world, the role of the gyroscope is played by a system with large spin. For example, Alice can send angular momentum eigenstates satisfying $\boldsymbol{n} \cdot \boldsymbol{J}|\psi\rangle = j|\psi\rangle$. This is essentially the solution proposed by Massar and Popescu [1] who took N parallel spins, polarized along $\boldsymbol{n}$. The fidelity of the transmission is usually defined as

$$F = \langle \cos^2(\chi/2)\rangle = (1 + \langle \cos\chi\rangle)/2\,, \qquad (1)$$

where χ is the angle between the true $\boldsymbol{n}$ and the direction indicated by Bob's measurement. The physical meaning of F is that $1 - F = \langle \sin^2(\chi/2)\rangle$ is the mean square error of the measurement, if the error is defined as $\sin(\chi/2)$. The experimenter's aim, minimizing the mean square error, is the same as maximizing fidelity. We can of course define "error" in a different way, and then fidelity becomes a different function of χ and optimization leads to different results [2]. Here, we shall take Eq. (1) as the definition of fidelity.

Massar and Popescu showed that for parallel spins, $1 - F = 1/(N + 2)$. It then came as a surprise that for $N = 2$, parallel spins were not the optimal signal, and a slightly higher fidelity resulted from the use of opposite spins [3]. The intuitive reason given for this result was the use of a larger Hilbert space (four dimensions instead of three). This raises the question, what is the most efficient signal state for N spins, whose Hilbert space has 2^N dimensions? Will F approach 1 exponentially? In this Letter, we show that the optimal result is a quadratic approach, as illustrated in Fig. 1.

Our first task is to devise Bob's measuring method, whose mathematical representation is a positive operator-valued measure (POVM) [4]. For any unit vector $\boldsymbol{n}$, not necessarily Alice's direction, let $|j, m(\boldsymbol{n})\rangle \equiv |j, m(\theta, \phi)\rangle$ denote the coherent angular momentum state [5] that satisfies

$$\boldsymbol{J}^2|j, m(\boldsymbol{n})\rangle = j(j + 1)|j, m(\boldsymbol{n})\rangle\,, \qquad (2)$$

and

$$\boldsymbol{n} \cdot \boldsymbol{J}|j, m(\boldsymbol{n})\rangle = m|j, m(\boldsymbol{n})\rangle\,. \qquad (3)$$

We then have [5]

$$(2j + 1)\int d_{\theta\phi}\,|j, m(\theta, \phi)\rangle\langle j, m(\theta, \phi)| = \mathbf{1}_j\,, \qquad (4)$$

where

$$d_{\theta\phi} := \sin\theta d\theta d\phi/4\pi\,, \qquad (5)$$

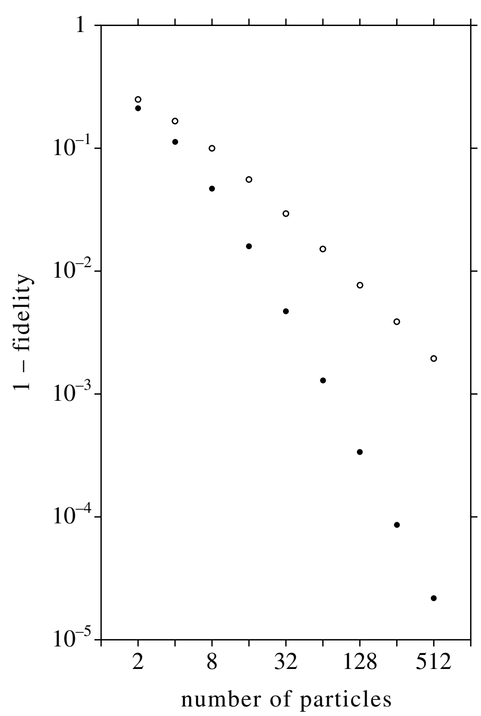

FIG. 1. $(1 - F)$ as a function of N. Open circles are for $m = j$ (Ref. [1]); closed circles are for $m = 0$ (this work).

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and $\mathbf{1}_j$ is the projection operator over the $(2j+1)$-dimensional subspace spanned by the vectors $|j, m(\theta, \phi)\rangle$. If $N = 2$, so that j is 0 or 1, the two resulting subspaces span the whole 4-dimensional Hilbert space. For higher N, all the rotation group representations with $j < N/2$ occur more than once. We then have, if we take each j only once, from 0 or $\frac{1}{2}$ to $N/2$,

$$\sum(2j+1) = \frac{(N+2)^2}{4} \quad \text{or} \quad \frac{(N+1)(N+3)}{4}, \quad (6)$$

for even or odd N, respectively. For large N, the dimensionality of the accessible Hilbert space tends to $N^2/4$, and this appears to be the reason that the optimal result for $1 - F$ is quadratic in N, not exponential. An intuitive argument for this quadratic behavior was given by Aharonov and Popescu [6]. No improvement results if we endow the particles with internal quantum numbers such as charge or strangeness, so that the entire Hilbert space can be spanned by states with distinguishable properties, because any additional information that Alice could send to Bob would refer to these new quantum numbers, not to the direction of $\boldsymbol{n}$.

We now turn to the construction of Bob's POVM [4]. Let ρ denote the initial state of the physical system that is measured. All these input states span a subspace of Hilbert space. Let $\mathbf{1}$ denote the projection operator on that subspace. A POVM is a set of positive operators E_μ which sum up to $\mathbf{1}$. The index μ is just a label for the outcome of the measuring process. The probability of outcome μ is $\mathrm{tr}(\rho E_\mu)$. In the present case, μ stands for the pair of angles $\theta\phi$ that are indicated by Bob's measurement. If we want a high accuracy, these output angles should have many different values, spread over the unit sphere [7]. For example, the components of a continuous POVM, as in Eq. (4), are given by

$$E_{\theta\phi} = (2j+1) d_{\theta\phi} |j, m(\theta, \phi)\rangle\langle j, m(\theta, \phi)|. \quad (7)$$

Such a POVM with $m = j$ corresponds to the method of Ref. [1]. The choice $m = j$ is not optimal. As shown in [3] for the case $N = 2$, signal states with opposite spins give a higher fidelity. With our present notations, these states are $[|0,0\rangle + |1, 0(\boldsymbol{n})\rangle]/\sqrt{2}$. They involve two values of j, but a single value of m, namely, 0.

One possibility to include several values of j in a POVM is to take a sum of expressions such as (4). This brings no advantage, because a convex combination of POVMs cannot yield more information than the best one of them [8]. Optimal POVM components can always be assumed to have rank one. Therefore each one of them should include all relevant j:

$$E_{\theta\phi} := d_{\theta\phi} |\theta, \phi\rangle\langle\theta, \phi|, \quad (8)$$

where

$$|\theta, \phi\rangle := \sum_{j=m}^{N/2} \sqrt{2j+1}\, |j, m(\theta, \phi)\rangle. \quad (9)$$

To verify that this is indeed a POVM, we note that in $\int E_{\theta\phi}$ there are diagonal terms $(2j+1)|j, m(\theta,\phi)\rangle \times \langle j, m(\theta, \phi)|$, which give $\mathbf{1}_j$, owing to Eq. (4). The off-diagonal terms with $j_1 \neq j_2$ vanish, as can be seen by taking their matrix elements between $\langle j_1, m_1|$ and $|j_2, m_2\rangle$ in the standard basis where J_z is diagonal [9]. We have [10]

$$\langle j_2, m(\theta, \phi) | j_2, m_2\rangle = \mathcal{D}^{(j_2)}_{mm_2}(\psi\theta\phi), \quad (10)$$

with a similar (complex conjugate) expression for $\langle j_1, m_1 | j_1, m(\theta, \phi)\rangle$. The rotation matrices $\mathcal{D}$ are explicitly given by

$$\mathcal{D}^{(j_2)}_{mm_2}(\psi\theta\phi) = e^{im\psi} d^{(j_2)}_{mm_2}(\theta) e^{im_2\phi}, \quad (11)$$

where the Euler angle ψ is related to an arbitrary phase which is implicit in the definition of $|j, m(\theta, \phi)\rangle$. Note that a single value of m occurs in all the components of the vectors $|\theta, \phi\rangle$ in Eq. (9), so that the undefined phases $e^{\pm im\psi}$ mutually cancel. It then follows from Eq. (4.6.1) of Ref. [10] that all the off-diagonal matrix elements of $\int E_{\theta\phi}$ vanish, so that we indeed have a POVM [11].

While Bob's optimal POVM is essentially unique in the Hilbert space that we have chosen, Alice's signal state, which is

$$|A\rangle = \sum_{j=m}^{N/2} c_j |j, m(\boldsymbol{n})\rangle, \quad (12)$$

contains unknown coefficients c_j. The latter are normalized,

$$\sum_{j=m}^{N/2} |c_j|^2 = 1, \quad (13)$$

but still have to be optimized.

The probability of detection of the pair of angles $\theta\phi$, indicated by the POVM component $E_{\theta\phi}$, is

$$\langle A|E_{\theta\phi}|A\rangle = d_{\theta\phi} \left| \sum_{j=m}^{N/2} c_j \sqrt{2j+1} \times \langle j, m(\theta, \phi) | j, m(\boldsymbol{n})\rangle \right|^2. \quad (14)$$

We have [5]

$$\langle j, m(\theta, \phi) | j, m(\boldsymbol{n})\rangle = e^{i\eta} d^{(j)}_{mm}(\chi), \quad (15)$$

where χ is the angle between the directions $\boldsymbol{n}$ and $\theta\phi$, and the phase $e^{i\eta}$ is related to the arbitrary phases which are implicit in the definitions of the state vectors in (15). The important point is that $e^{i\eta}$ does not depend on j and therefore is eliminated when we take the absolute value of the sum in Eq. (14). Explicitly, we have

$$d^{(j)}_{mm}(\chi) = \cos^{2m}(\chi/2) P^{(0,2m)}_{j-m}(\cos\chi), \quad (16)$$

where $P^{(a,b)}_n(x)$ is a Jacobi polynomial [5,10]. We shall write $x = \cos\chi$ for brevity, so that the fidelity is

$$F = (1 + \langle x\rangle)/2. \quad (17)$$

Our problem is to find the coefficients c_j that maximize $\langle x \rangle$. Owing to rotational symmetry, we can assume that Alice's direction $\boldsymbol{n}$ points toward the z axis, so that $d_{\theta\phi}$ can be replaced by $dx/2$ after having performed the integration over ϕ. We thus obtain

$$\langle x \rangle = \frac{1}{2} \int_{-1}^{1} x\, dx \left| \sum_{j=m}^{N/2} c_j \sqrt{2j+1} \left(\frac{1+x}{2} \right)^m \times P_{j-m}^{(0,2m)}(x) \right|^2 . \tag{18}$$

This integral can be evaluated explicitly by using the orthogonality and recurrence relations for Jacobi polynomials [12,13]. The result is

$$\langle x \rangle = \sum_{j,k} c_j^* c_k A_{jk} , \tag{19}$$

where A_{jk} is a real symmetric matrix, whose only nonvanishing elements are

$$A_{jj} = m^2/[j(j+1)], \tag{20}$$

and

$$A_{j,j-1} = A_{j-1,j} = (j^2 - m^2)/j\sqrt{4j^2 - 1} . \tag{21}$$

The optimal coefficients c_j are the components of the eigenvector of A_{jk} that corresponds to the largest eigenvalue, and the latter is $\langle x \rangle$ itself. The result of the calculation is displayed in Fig. 1 for $m = 0$ (which is best) and $m = j$ (which is the method investigated in Ref. [1]). For $m = 0$ and large N, we find that

$$1 - F \rightarrow [2.4048/(N+3)]^2, \tag{22}$$

where the numerator is the first zero of the Bessel function J_0 [14]. The right-hand side ought to be compared to the result of [1], which was $1/(N+2)$.

For $N = 2$ and $m = 0$, our result coincides with Ref. [3]. For $N = 3$, we obtain $F = 0.844\,95$ with $c_{3/2} = 0.603\,62$ and $c_{1/2} = 0.797\,55$. The results for larger N and intermediate values of m gradually fall between those displayed in Fig. 1. Had we chosen a definition of fidelity other than Eq. (1), these results would of course be different, but the method for solving the problem is in principle the same.

It thus appears that it is advantageous to take the lowest possible m (namely, $m = 0$ for even N and $m = \frac{1}{2}$ for odd N). This is intuitively quite plausible [6]. Our numerical results agree with the analytical treatment in Ref. [14], whose details appeared after the present Letter was submitted. We are grateful to the authors of Ref. [14] for clarifying their work.

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[7] If $N \leq 4$, an optimal POVM can have a finite number of components, corresponding to the vertices of regular polyhedra. In general, a POVM with a nonuniform distribution of directions on the unit sphere can be converted into a uniform one by rotational symmetrization. This does not affect the fidelity of transmission if the input is uniform. We can therefore assume that the POVM distribution is uniform without loss of generality.
[8] A convex combination of POVMs, such as $E_\mu = \sum w_k E_{k\mu}$, with $w_k > 0$ and $\sum w_k = 1$, has the physical meaning that each one of the sets $\{E_{k\mu}\}$ is chosen by the experimenter with probability w_k.
[9] This is the reason each value of j can occur only once in the POVM, as otherwise there would be off-diagonal elements in the sum. Such off-diagonal elements can still be avoided by introducing new "internal" quantum numbers but, as explained above, any additional information would refer to these new numbers, not to the direction indicated by Alice.
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