Finite-Mode Spectral Model of Homogeneous and Isotropic Navier-Stokes Turbulence: A Rapidly Depleted Energy Cascade

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An eddy-viscous term is added to Navier-Stokes dynamics at wave numbers k greater than the inflection point k_c of the energy flux $F(\log(k))$. The eddy viscosity is fixed so that the energy spectrum satisfies $E(k) = E(k_c)(k/k_c)^{-3}$ for $k > k_c$. This resulting forcing induces a rapid depletion of the energy cascade at $k > k_c$. It is observed numerically that the model reproduces turbulence energetics at $k \leq k_c$ and statistics of two-point velocity correlations at scales $r > \lambda$ (Taylor microscale). Compared to a direct numerical simulation of $R_\lambda = 130$ an equivalent run with the present model results in a gain of a factor 20 in CPU time.

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In the limit of large Reynolds numbers, Navier-Stokes (NS) turbulence exhibits an infinite number of excited modes (scales of motions), each subject to nontrivial phase mixing induced by the quadratic nonlinearities inherent to hydrodynamics. A key issue lies in the reduction of the dynamics to a closed set of equations for a finite number of modes. The low wave-number modes, or large-scale motions, are particularly important since they contain most information about the energetics.

We present a reduced set of dynamical equations governing the energy-containing modes of homogeneous and isotropic turbulence.

As a paradigm for homogeneous and isotropic turbulence we consider the incompressible NS equations in a cyclic box of side length 2π [1]. The velocity field may be expanded as a Fourier series $\boldsymbol{v}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \boldsymbol{v}(\boldsymbol{k}, t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x})$, so that the NS equations in Fourier space read

$$\left[\frac{\partial}{\partial t} + \nu_{\mathrm{mol}} k^2\right] v_\alpha(\boldsymbol{k}, t) = -ik_\gamma \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} v_\beta(\boldsymbol{p}, t) v_\gamma(\boldsymbol{q}, t) + f_\alpha(k, t). \quad (1)$$

Summation over repeated Greek indices is implied. In (1), ν_{mol} is the molecular kinematic viscosity, while $f_\alpha(k, t)$ is an isotropic force acting at very low k.

Under isotropic conditions the wave-number–by–wave-number instantaneous spectral energy budget is

$$\left[\frac{\partial}{\partial t} + 2\nu_{\mathrm{mol}} k^2\right] E(k, t) = T(k, t) + \epsilon_{\mathrm{inj}}(k, t).$$

$E(k, t)$ denotes the energy in the shell of wave vectors $k \leq |\boldsymbol{k}| < k + 1$, $T(k, t)$ is the energy-transfer spectrum comprising all triad interactions involving wave-number k modes, and $\epsilon_{\mathrm{inj}}(k, t)$ represents the rate of energy supplied by the large-scale force $\mathbf{f}(k, t)$.

A finite-mode spectral model deals with a truncated expansion of the velocity field, such that $\boldsymbol{v}(\boldsymbol{k}, t) = \mathbf{0}$ for $|\boldsymbol{k}| > K$ (spectral cutoff). At very large wave numbers, turbulent excitations are suppressed by viscous dissipation. Consequently, if K is chosen larger than the characteristic dissipative wave number K_{diss} the missing triad interactions in the NS system reduced to resolved modes may be neglected. The integration in time of this reduced system is known as a direct numerical simulation (DNS) of (1). Dimensional arguments suggest that $K_{\mathrm{diss}} \sim R_\lambda^{3/2}$ [2], where R_λ is the Taylor microscale Reynolds number. Because physical space is three dimensional, the number of retained modes scales as $R_\lambda^{9/2}$ and thus precludes the investigation of high-R_λ turbulent flows by DNS. In what follows we focus on case $K \ll K_{\mathrm{diss}}$ so as to achieve a substantial reduction of the NS system. A model accounting for missing triad interactions is then required.

Classical phenomenology pictures the kinetic-energy transfer mechanism of turbulence as an energy cascade from low to high wave-number modes [1]. The cascade may be characterized by the flux $F(k, t)$ of energy across wave number k in terms of which the local stationary energy budget may be written as

$$2\nu_{\mathrm{mol}} k^3 E(k) = -dF/d\log k. \quad (2)$$

When time dependence is omitted ensemble averaging is assumed. A peculiar feature of $F(k)$ as a function of $\log(k)$ lies in the observation of an inflection point at wave number k_c, i.e., $[d^2F(k)/d\log k^2]_{k=k_c} = 0$ [see Fig. 1]. In order to capture the physical meaning of k_c let us introduce the characteristic eddy-turnover time $t_{\mathrm{eddy}}(k)$, or inverse rate of energy transfer. Following Kolmogorov phenomenology [2] it may be estimated as

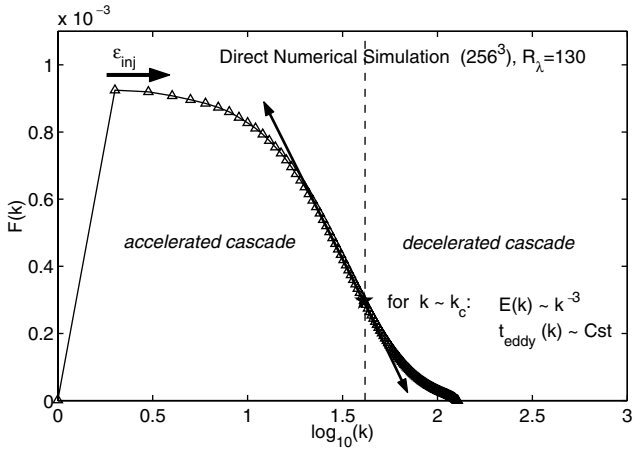

FIG. 1. $F(k)$ denotes the mean energy flux across k. As a function of $\log(k)$ the flux exhibits an inflection point at $k = k_c$. For $k < k_c$ the energy cascade accelerates with k, whereas for $k > k_c$ the cascade decelerates under predominant viscous effects.

$$t_{\mathrm{eddy}}(k) \sim E(k)^{-1/2} k^{-3/2}. \tag{3}$$

From (2) and (3) it is now seen that $t_{\mathrm{eddy}}(k)$ decreases, i.e., the energy transfer accelerates, with $k \ll k_c$; Kolmogorov's spectrum $E(k) \sim k^{-5/3}$ [2] yields $t_{\mathrm{eddy}}(k) \sim k^{-2/3}$. As k increases, viscous effects intensify and the cascade acceleration progressively vanishes. For $k \lesssim k_c$ the energy flows at a constant speed and $E(k) \sim k^{-3}$ locally. Finally for $k > k_c$ the cascade is decelerated and the energy flux slowly vanishes. Based on these observations we propose a finite-mode model obtained from adding an isotropic force $-\nu(k > k_c \,|\, K, t) k^2 \boldsymbol{v}(\boldsymbol{k}, t)$ to the (reduced) NS dynamics at $k_c < k \leq K$. As will be shown below the additional force accounts for a rapid depletion of the energy cascade in the range $k_c < k \leq K$ (near the cutoff) without perturbing the modes in the *accelerated cascade* range. Further, incompressibility and Galilean invariance of the NS equations are preserved. Note that only the amplitude of modes $k_c < k \leq K$ is rescaled, their phase remaining untouched. The model spectral energy budget reads

$$\left[\frac{\partial}{\partial t} + 2\nu_{\mathrm{eddy}}(k \,|\, K, t) k^2\right] E(k, t) = T(k \,|\, K, t) + \epsilon_{\mathrm{inj}}(k, t),$$

where $T(k \,|\, K, t)$ is the energy-transfer spectrum reduced to triad interactions among resolved modes. Following [3], $\nu_{\mathrm{eddy}}(k \,|\, K, t) \equiv \nu_{\mathrm{mol}} + \nu(k > k_c \,|\, K, t)$ may be viewed as a dynamic wave-number dependent eddy viscosity.

The variable component $\nu(k > k_c \,|\, K, t)$ of the eddy viscosity is adjusted so as to extrapolate the mean $E(k) \sim k^{-3}$ scaling in the damping zone, i.e.,

$$E(k) = E(k_c)\,(k/k_c)^{-3}, \qquad k_c < k \leq K. \tag{4}$$

The scaling $E(k) \sim k^{-3}$ is heuristically associated with the joined k independent enstrophy cascade and identically zero energy cascade in two-dimensional turbulence [4]. Here the enforced $E(k) \sim k^{-3}$ is expected to transport vorticity to $k > k_c$ and to rapidly deplete the energy flux. This mechanism is connected with the picture of locally two-dimensional fluid motion around regions of concentrated vorticity (coherent vorticity structures) displaying a brutal energy penalty as the fluid reaches the core of these regions. It is noteworthy that the scaling $E(k) \sim k^{-3}$ has been experimentally evidenced in the vicinity of a stable, strong 3D vortex whose surrounding fluid is brought to spiral rapidly towards the core of the vortex [5]. An important point to note here is the potential nonlocality of the cascade process in the damping range. Indeed, following Kraichnan's arguments [4], a k^{-3} spectrum is consistent with all triad interactions contributing equally (the same amount) to the energy transfer at k. It is therefore expected that the energy cascade (around k_c) no longer operates through eddy breaking but rather through locally coherent (in physical space) fluid motions. Recall that the unmodified phases of the forced modes in the model effectively allow for such coherence to be established [6].

Numerical experiments performed with a parallel distributed memory pseudospectral NS solver [7] are presented next. The large-scale kinetic-energy forcing is adjusted at each time step by scaling the amplitudes of modes $1.5 \leq k < 2.5$ uniformly (phases are left to fluctuate freely), so as to compensate exactly the losses due to eddy dissipation in the kinetic-energy budget. This robust and efficient forcing scheme permits a rapid relaxation to stationarity [8]. Results from DNS ($K > K_{\mathrm{diss}}$) and finite-mode model simulations ($K \ll K_{\mathrm{diss}}$) are compared using identical initial conditions and statistics sampled in space and time. Note that finite-mode simulations will be referred to as large-eddy simulations (LES) in figure captions.

In practice the cutoff wave number K is fixed and the molecular viscosity is chosen such that $k_c \simeq 2K/3$; the damping zone extends over $1/6$ of a decade. Figure 3 below shows that this width is sufficient for the completion of a rapid falloff of the energy flux. At each time step $\nu(k > k_c \,|\, K, t)$ is updated in order to enforce a power-law scaling of $E(k, t)$ in the damping zone; $E(k, t) = E(k_c, t)\,(k/k_c)^{-\alpha(t)}$ with $\alpha(t) = -[\partial \log E(k, t)/\partial \log k]_{k=k_c}$. This is to be viewed as a linear extrapolation in log - log coordinates. An alternative adjustment scheme consists in determining $k_c(t)$ at each time step and extrapolating $E(k, t) = E[k_c(t), t][k/k_c(t)]^{-3}$ for $k > k_c(t)$. Both methods have been tested and yield identical results. For convenience the first one has been retained. The derivative is estimated by a least-squares approximation. Comparisons between aliased and dealiased (using the zero padding) schemes have shown that the eddy viscosity suitably damps aliasing errors, mainly concentrated in near wave number K shells.

The normalized mean kinetic-energy and squared-vorticity spectra [9] are shown in Fig. 2. They collapse in the whole energy-containing range, indicating that

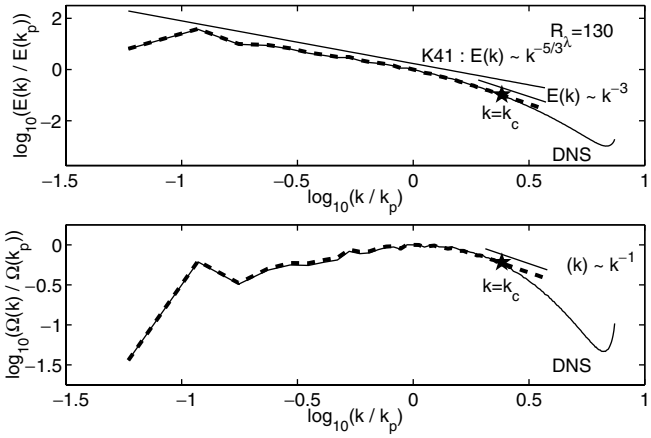

FIG. 2. Energy $E(k)$ and squared-vorticity $\Omega(k)$ spectra from direct and finite-mode numerical simulations. The spectra collapse when normalized by the wave number k_p of maximum squared-vorticity spectrum and spectrum amplitude at k_p [9].

our closure scheme does not notably affect energetics at wave numbers $k \leq k_c$. Normalized mean energy fluxes for various Reynolds numbers collapse in Fig. 3. As expected, energy fluxes rapidly falloff and vanish in the damping zone.

An *effective* eddy viscosity is defined by $\overline{\nu}_{\rm eddy}(k \mid K) = -T(k \mid K)/2k^2E(k)$. In Fig. 4, $\overline{\nu}_{\rm eddy}(k \mid K)$ exhibits a cusp near K to compensate missing energy exchanges with unresolved modes and to avoid a piling up of energy at the bottom of the cascade. Interestingly all eddy viscosities collapse, showing some degree of self-similarity in the energetics of the rapid depletion process. Note that this self-similar form is expected to depend on the ratio k_c/K. As $k \to {k_c}^+$, $\overline{\nu}_{\rm eddy}(k \mid K) \to \nu_{\rm mol}$ smoothly, suggesting that $k \lesssim k_c$ modes are suitably synchronized with $k \geq k_c$ modes.

Consider now the two-point correlations of the unfiltered velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. All Fourier modes have been retained in order to examine small-scale effects of our high

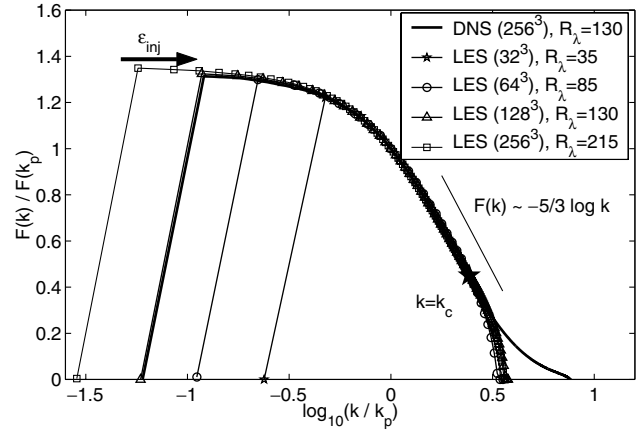

FIG. 3. Normalized mean kinetic-energy fluxes, from direct (DNS) and finite-mode (LES) numerical simulations, collapse in the energy-containing range of wave numbers.

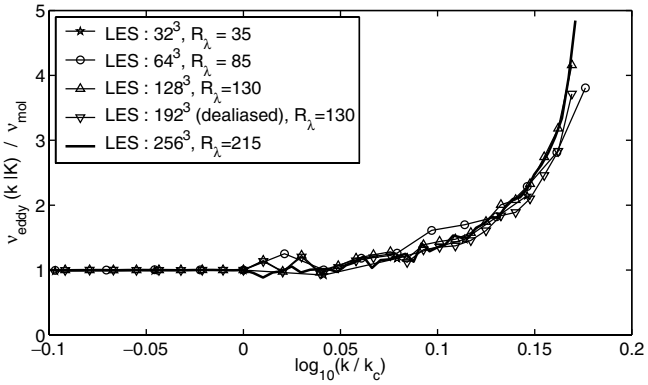

FIG. 4. The effective viscosity $\overline{\nu}_{\rm eddy}(k \mid K)$ is displayed for different LES simulations. As $k \to k_c$, $\overline{\nu}_{\rm eddy}(k, K) \to \nu_{\rm mol}$ smoothly. Oscillations for $k \geq k_c$, at low resolution, result from the discrete wave-number decomposition.

wave-number damping. We focus first on moments of the longitudinal velocity increments $\delta_\parallel v(r)$ across a distance r, also called structure functions [1]. An exact relation, the Howarth-VonKarman equation [1], involving the second- and third-order structure functions at small scales, follows from the NS equations:

$$\langle \delta v_\parallel(r)^3 \rangle = -4/5 \varepsilon_{\rm inj} r + 6\nu d\langle \delta v_\parallel(r)^2 \rangle / dr \, .$$

This equation is verified in Fig. 5. The probability density functions of $\delta_\parallel v(r)$ are compared in Fig. 6. The agreement is satisfactory.

Relative scalings of velocity structure functions have recently received much interest [10,11]. The (local) intermittency exponent $\mu(r)$ is given by

$$2 + \mu(r) = d \log\langle |\delta v_\parallel(r)|^6 \rangle / d \log\langle |\delta v_\parallel(r)|^3 \rangle \, .$$

Results displayed in Fig. 7 are consistent with *effective* values $\bar{\mu} \lesssim -0.2$ as reported in [11]. At $R_\lambda = 130$, we observe that all estimates of $\mu(r)$ almost coincide at scales $r > \lambda$, the Taylor microscale [1]. Discrepancies at $r \leq \lambda$ are clearly related to our rapid depletion procedure. The Taylor microscale is here used as a reference scale and

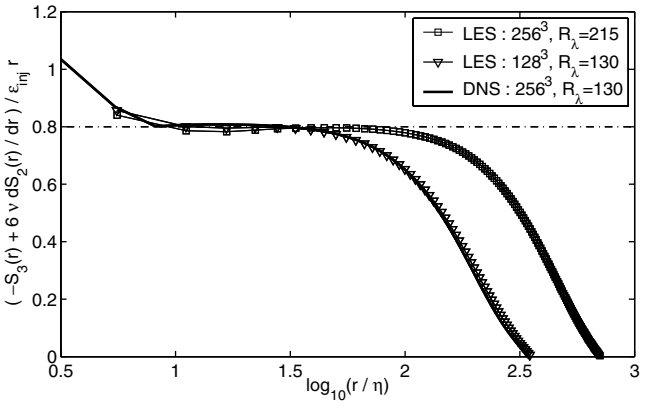

FIG. 5. The Howarth-VonKarman equation is tested. The Kolmogorov's dissipative scale $\eta \equiv (\nu_{\rm mol}^3/\varepsilon_{\rm inj})^{1/4}$. Note the linear vertical axis.

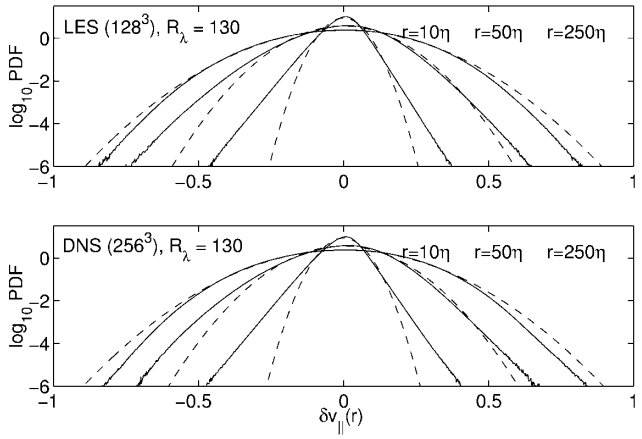

FIG. 6. Distributions of longitudinal velocity increments, at various scales, collapse. Small discrepancies may be due to the two sets of compared scales not being exactly identical.

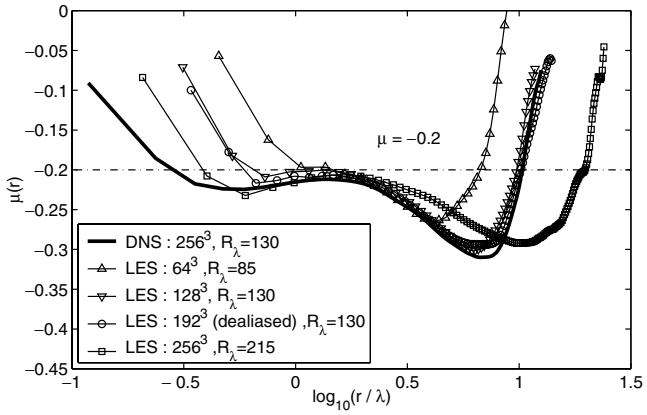

FIG. 7. The intermittency exponent $\mu(r)$. The scale is normalized by the Taylor microscale $\lambda = \sqrt{15\nu v_{\rm rms}^2/\varepsilon_{\rm inj}}$.

should not be taken as a strict lower bound for our model. We would like to stress that $\mu(r)$ is a sharp (local) estimator of turbulence scaling properties. Finally, the numerical tests strongly suggest that both energetics and scaling properties of the energy-containing turbulent fluctuations are well reproduced by this model.

The present rapidly depleted energy cascade model is of practical interest as it allows us to investigate the statistics at scales $r > \lambda$, without resolving the full NS system. A factor 20 gain in CPU time compared to a DNS at $R_\lambda = 130$ is obtained. More importantly we may reach $R_\lambda \simeq 550$ in a 1024^3 LES. In virtue of the relatively extended $F(k) \sim \log(k)$ scaling (see Fig. 1) the closure is not very sensitive to the exact values of $(K, k_c, \nu_{\rm mol})$. From a low-resolution simulation higher-resolution runs can be directly set up by simply rescaling the parameters $K \to aK$, $k_c \to ak_c$, and $\nu_{\rm mol} \to a^{-4/3}\nu_{\rm mol}$. However, the model is of theoretical interest also, because understanding the inflection point of the kinetic-energy flux and recasting the forced dynamics into the framework of nonequilibrium stationary states [12] are open and important problems.

Molecular viscosity acting at low wave numbers is essential in that it progressively slows the energy cascade. Our model has shown that it is indeed possible to rapidly "thermostate" the cascade dynamics [12] from $k \sim k_c$ onwards when the cascade finally reaches a constant flow rate. Note that *thermostating* the energy cascade in the acceleration range, i.e., at $k \ll k_c$, produces strong backward perturbations on unforced modes. The nonlocality of triad interactions around k_c is thought to be responsible for the adequate phase synchronization (coupling) of *thermostated* modes with unforced modes and could provide a clue to the relevance of this model. Finally, including true viscosity at low wave numbers was already suggested in [3,6], where it was demonstrated that *proper* dissipation must act on low wave-number modes to be consistent with an energy cascade displaying an $E(k) \sim k^{-5/3}$ spectrum.

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[1] U. Frisch, *Turbulence: the Legacy of Kolmogorov* (Cambridge University Press, Cambridge, UK, 1995).
[2] A. N. Kolmogorov, C. R. Acad. Sci. USSR **30**, 301 (1941).
[3] R. H. Kraichnan, J. Atmos. Sci. **33**, 1521 (1976).
[4] R. H. Kraichnan, Phys. Fluids **33**(7), 1417 (1967).
[5] F. Chilla and J.-F. Pinton, in *Advances in Turbulence VII,* edited by U. Frisch (Kluwer Academic, Dordrecht, 1998), pp. 211–214.
[6] Z.-S. She and E. Jackson, Phys. Rev. Lett. **70**, 1255 (1993).
[7] C. R. Koudella, Ph.D. thesis, Ecole Normale Supérieure de Lyon (France), 1999.
[8] V. Eswaran and S. B. Pope, Comput. Fluids **16**(3), 257 (1998).
[9] Z.-S. She *et al.,* Phys. Rev. Lett. **70**, 3251 (1993).
[10] R. Benzi *et al.,* Phys. Rev. E **48**, 29 (1993).
[11] N. Cao, S. Chen, and Z.-S. She, Phys. Rev. Lett. **76**, 3711 (1996).
[12] G. Gallavotti, Physica (Amsterdam) **105D**, 163 (1997).