# Implications of the ALEPH $\tau$-Lepton Decay Data for Perturbative and Nonperturbative QCD

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We use ALEPH data on hadronic $\tau$ decays in order to calculate Euclidean coordinate space correlation functions in the vector and axial-vector channels. The linear combination $V - A$ receives no perturbative contribution and is quantitatively reproduced by the instanton liquid model. In the case of $V + A$ the instanton calculation is in good agreement with the data once perturbative corrections are included. These corrections clearly show the evolution of $\alpha_s$. We also analyze the range of validity of the operator product expansion (OPE). We conclude that the range of validity of the OPE is limited to $x \lesssim 0.3$ fm, whereas the instanton model describes the data over the entire range.

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Quantitative understanding of the interface between perturbative and nonperturbative effects is the central problem in QCD dynamics. Historically, QCD sum rules based on the operator product expansion (OPE) [1,2] constituted the first serious attempt to describe nonperturbative phenomena in QCD. The initial application of QCD sum rules to vector and axial vector meson leads to very promising results. It was soon discovered, however, that not all hadrons are alike [3]. Phenomenology demands that nonperturbative effects in scalar and pseudoscalar channels, both meson and glueball, are much bigger than in the vector channels. This fact is not reproduced by the OPE but it was realized that direct instanton effects appear in exactly those channels in which nonperturbative effects are large. This observation gave rise to the instanton liquid model [4].

The available information on hadronic correlation functions, from experimental data, the OPE, and other exact results, was reviewed in [5]. Since then, the high statistics measurement of hadronic $\tau$ decays, $\tau \to \nu_\tau$ + hadrons, by the ALEPH experiment at CERN [6] has significantly improved the experimental situation in the vector and axial-vector channel. The purpose of this paper is to compare these results with theoretical predictions, from both the OPE and the instanton models. In particular, we would like to assess the range of applicability of the two approaches and put improved constraints on the parameters that enter into the calculations. Translating the spectral functions measured by the ALEPH Collaboration into Euclidean coordinate space correlation functions will also allow precise comparison of the experimental data with improved lattice calculations along the lines of [7].

In the following, we shall consider the vector and axial-vector correlation functions $\Pi_V(x) = \langle j^a_\mu(x) j^a_\mu(0)\rangle$ and $\Pi_A(x) = \langle j^{5a}_\mu(x) j^{5a}_\mu(0)\rangle$. Here, $j^a_\mu(x) = \bar q \gamma_\mu \frac{\tau^a}{2} q$, and $j^{5a}_\mu(x) = \bar q \gamma_\mu \gamma_5 \frac{\tau^a}{2} q$ are the isotriplet vector and axial-vector currents. The correlation functions have the spectral representation [5]

$$\Pi_{V,A}(x) = \int ds\, \rho_{V,A}(s) D(\sqrt{s}, x), \qquad (1)$$

where $D(m,x) = m/(4\pi^2 x)K_1(mx)$ is the Euclidean coordinate space propagator of a scalar particle with mass $m$. We will focus on the linear combinations $\Pi_V + \Pi_A$ and $\Pi_V - \Pi_A$. These combinations allow for a clearer separation of different nonperturbative effects. The corresponding spectral functions $\rho_V \pm \rho_A$ which are measured by the ALEPH Collaboration are shown in Fig. 1.

In QCD, the vector and axial-vector spectral functions have to satisfy chiral sum rules. If we assume that $\rho_V(s) - \rho_A(s) = 0$ above the maximum invariant mass $s = m_\tau^2$ for which the spectral functions can be measured, we find that the ALEPH data satisfy all chiral sum rules within the experimental uncertainty. However, the central values of the sum rules differ significantly from the chiral predictions [6]. In general, both $\rho_V$ and $\rho_A$ are expected to show oscillations of decreasing amplitude [2]. If we set $\rho_V(s) - \rho_A(s) = 0$ above an arbitrarily chosen invariant mass $s_0$, this will lead to the appearance of spurious

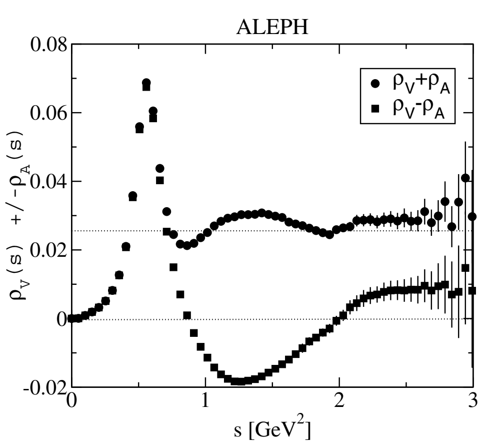


FIG. 1. Spectral functions $\rho_V(s) + \rho_A(s)$ extracted by the ALEPH Collaboration. The dotted lines show the asymptotic (free field) limit.

dimension $d = 2, 4$ operators in the correlation functions at small $x$. For this reason we decided to use $s_0 = 2.5\ \mathrm{GeV}^2$, which is slightly below the tau mass but allows all chiral sum rules to be satisfied. The reader should be aware of the fact that we have, in effect, slightly moved the data points in the small $x$ region within the error bars reported by the ALEPH Collaboration. Finally we add the pion pole contribution, which is not shown in Fig. 1, to the axial-vector spectral function. This corresponds to an extra term $\Pi_A^\pi(x) = f_\pi^2 m_\pi^2 D(m_\pi, x)$. The resulting correlation functions $\Pi_V(x) \pm \Pi_A(x)$ are shown in Fig. 2.

We begin our analysis with the combination $\Pi_V - \Pi_A$. This combination is sensitive to chiral symmetry breaking, while perturbative diagrams, as well as gluonic operators, cancel.

In Fig. 2 we compare the measured correlators to predictions from the instanton model. These predictions are described in [8], where we also discuss results for the $\rho, a_1$ masses and couplings. The main assumption of the instanton model is that the QCD vacuum is dominated by strong nonperturbative field configurations, instantons. In the simplest version, the random instanton liquid, the instanton positions, and color orientations are distributed randomly. The ensemble is characterized by two numbers, the instanton density $(N/V) = 1\ \mathrm{fm}^{-4}$ and the average instanton size $\rho = 1/3$ fm. These parameters were fixed a long time ago by using the requirement that they must reproduce the phenomenological values of the quark and gluon condensates [4]. The agreement of the instanton prediction with the measured $V - A$ correlation is impressive and extends all the way from short to large distances. At distances $x > 1.25$ fm both combinations are dominated

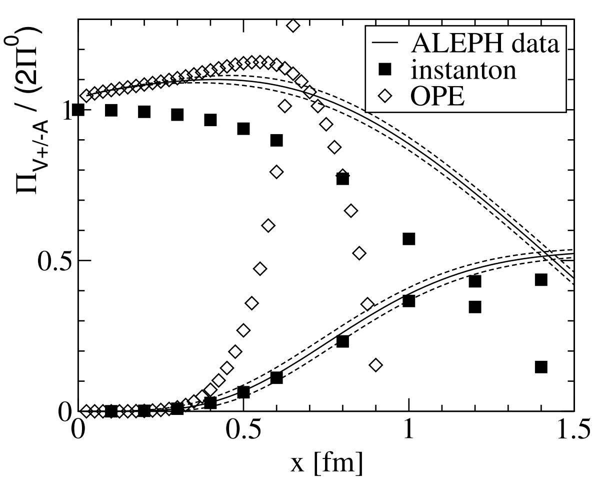


FIG. 2. Euclidean coordinate space correlation functions $\Pi_V(x) \pm \Pi_A(x)$ normalized to free field behavior. The solid lines show the correlation functions reconstructed from the ALEPH spectral functions and the dashed lines are the corresponding error band. The squares show the result of a random instanton liquid model and the diamonds show the OPE fit described in the text.

by the pion contribution while at intermediate $x$ the $\rho, \rho'$ and $a_1$ resonances contribute.

In order to study the validity of the operator product expansion we have to study the short distance region in more detail. The OPE predicts that the $V - A$ correlation function starts with the following quark-antiquark operators of dimensions $d = 4$ and $d = 6$,

$$\frac{\Pi_V(x) - \Pi_A(x)}{2\Pi_0(x)} = -\frac{\pi^2}{4} m\langle\bar{q}q\rangle x^4 + \frac{\pi^3}{9}\alpha_s(x)\langle\bar{q}q\rangle^2 \log(x^2) x^6 + \dots . \tag{2}$$

We have used the factorization assumption to simplify the $\langle(\bar{q}q)^2\rangle$ operator. In general, factorization fails badly in the instanton model, but it is a good approximation for the particular operators that appear in $\Pi_{V,A}$. The value of the dimension $d = 4$ operator is determined by the Gell-Mann-Oakes-Renner (GMOR) relation to be $(x/1.66\ \mathrm{fm})^4$. By using $\langle\bar{q}q\rangle = -(230\ \mathrm{MeV})^3$ and the one-loop running coupling constant we also estimate the $d = 6$ operator as $(x/0.66\ \mathrm{fm})^6$. This implies that the $d = 6$ operator totally dominates over the $d = 4$ operator.

This estimate can be checked by considering the value of the $d = 6$ operator as a free parameter and trying to extract it from the measured data. Because the $d = 4$ operator is so small, we use the GMOR value. A similar determination of power corrections was already done by the ALEPH Collaboration using moment sum rules [6] (see also [9–11]). Nevertheless, fitting the OPE coefficients into coordinate space provides important additional insight. The results depend on the coordinate range $[0, x_m]$ used in the fit, but for $x_m < 0.3$ fm this dependence is weak. The result for $x_m = 0.3$ fm, also shown in Figs. 2 and 3, is

$$\frac{\Pi_V(x) - \Pi_A(x)}{2\Pi_0(x)} = \left(\frac{x}{1.66\ \mathrm{fm}}\right)^4 + \left(\frac{x}{0.66\ \mathrm{fm}}\right)^6 + \dots . \tag{3}$$

We find that the size of the dimension $d = 6$ term agrees with the SVZ (Shifman, Vainshtein, and Zakharov) prediction. However, the range of convergence of the OPE is only $x \lesssim 0.3$ fm. We also note that the accuracy of the data in this regime is very poor.

We can also check the short distance behavior of the correlators in the instanton liquid. Instantons generate the same $d = 4$ operator as the OPE but the nature of the $d = 6$ operator is different. To leading order in the semiclassical expansion there is no radiatively generated $\alpha_s\langle\bar{q}q\rangle^2 \log(x^2) x^6$ operator, but instead there is a nonsingular $\langle\bar{q}q\rangle^2 x^6$ term. Such terms are dropped in the standard OPE, but they are present in the coordinate space correlators. The numerical value of this term is $(x/0.64\ \mathrm{fm})^2$, close to the data and the OPE term.

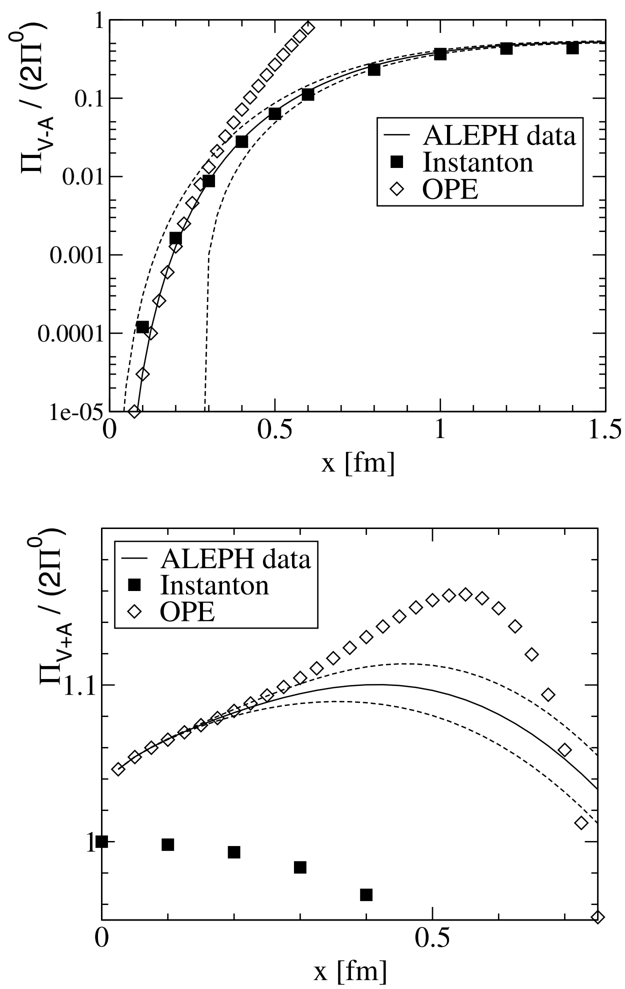


FIG. 3. Same as Fig. 2, but with the $\Pi_V(x) - \Pi_A(x)$ correlator plotted on a logarithmic scale (top) and $\Pi_V(x) + \Pi_A(x)$ shown in more detail (bottom).

We now focus our attention on the $V + A$ correlation function. The unique feature of this function is that the full correlator is close to the free field result for distances as large as 1 fm. This phenomenon was referred to as "super-duality" in [5].

The instanton model reproduces this feature of the $V + A$ correlator. We also notice that for small $x$ the deviation of the correlator in the instanton model from free field behavior is small compared to the perturbative $O(\alpha_s/\pi)$ correction. This opens the possibility of precision studies of the pertubative QCD (pQCD) contribution. But before we do so, let us compare the correlation functions to the OPE prediction

$$\frac{\Pi_V(x) + \Pi_A(x)}{2\Pi_0(x)} = 1 + \frac{\alpha_s}{\pi} - \frac{1}{384}\langle g^2 (G^a_{\mu\nu})^2 \rangle x^4 - \frac{2\pi^3}{81}\alpha_s(x)\langle \bar{q}q \rangle \log(x^2) x^6 + \dots . \quad (4)$$

Note that the perturbative correction is attractive, while the power corrections of dimensions $d = 4$ and $d = 6$ are repulsive. Direct instantons also induce an $O(x^4)$ correction $1 - \frac{\pi^2}{12}(\frac{N}{V})x^4 + \dots$ [12–14], which is consistent with the OPE because in a dilute instanton liquid we have $\langle g^2 G^2 \rangle = 32\pi^2 (N/V)$. This term can indeed be seen in the instanton calculation and causes the correlator to drop below 1 at small $x$.

It is possible to extract the value of $\Lambda_{\rm QCD}$ together with the power corrections from the data. Because the dimension-6 operator is relatively small we have fixed it from a joint fit with the $V - A$ correlator. We find

$$\frac{\Pi_V(x) + \Pi_A(x)}{2\Pi_0(x)} = 1 + \frac{\alpha_s(x)}{\pi} - \left(\frac{x}{1.52\ \text{fm}}\right)^4 - \left(\frac{x}{0.85\ \text{fm}}\right)^6 + \dots \quad (5)$$

and $\alpha_s(m_\tau) \simeq 0.35$ [15], which is consistent with other determinations [6]. The value of the gluon condensate term is quite uncertain but comes out smaller than the SVZ estimate $-(x/1.0\ \text{fm})^4$ [1]. This estimate suggests that the power corrections cancel the pQCD contribution at about $x \sim 0.5$ fm, which is not seen in the data. The most likely conclusion is that the range of validity of the OPE is $x \lesssim 0.3$ fm, and that the power corrections are always small when compared to the pQCD term. In that case, however, it is doubtful that one will ever be able to extract the value of the gluon condensate.

Finally, we address the purely perturbative contribution to the $V + A$ correlation function, using the instanton calculation as a representation of the nonperturbative part of the correlation function. This is supported by the fact that instantons provide a very accurate description of the $V - A$ correlator which is free of perturbative contributions. The difference between the full correlation function and the instanton calculation is shown by the squares in Fig. 4. For comparison, we also show the full correlation function with only the free field behavior subtracted. At a short distance, there is no difference between the two curves, and both follow the first order perturbative result $\alpha_s(x)/\pi$. At larger distances $\Pi_V(x) + \Pi_A(x) - 2\Pi^0(x)$ starts to drop, but the nonperturbatively subtracted correlator continues to grow. This behavior nicely shows the running of $\alpha_s$ even at moderately large $x$ [16]. At $x \lesssim 0.3$ fm the agreement becomes even better if the two-loop contribution is added, but in this case the Landau pole is reached earlier. For this reason, the good agreement of the data with the one-loop result even for large $x > 0.3$ fm may be somewhat coincidental. The reason one is able to follow the pQCD behavior well outside the usual perturbative domain is the remarkable degree of cancellation among nonperturbative effects. Further high statistics studies of this issue using lattice simulations would be very interesting.

In summary, we have used the high statistics ALEPH data on hadronic $\tau$ decays to calculate Euclidean space

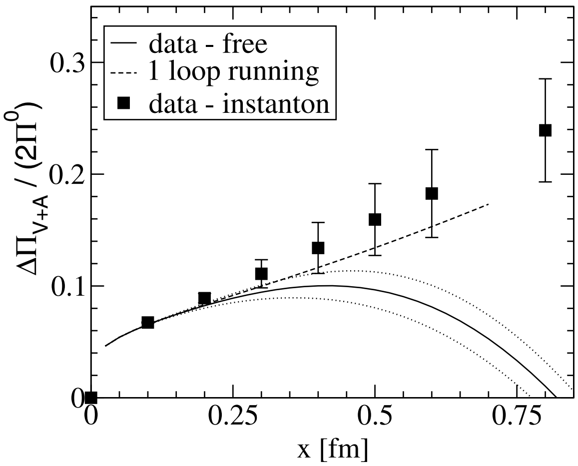


FIG. 4. This figure shows an estimate of the perturbative part of the $V + A$ correlation function. The solid line is the measured correlation function with the free field correlator subtracted. The squares show the measured correlator with the instanton contribution subtracted and the dashed line is the one-loop prediction.

correlation functions in the vector and axial-vector channels. We focused our discussion on the linear combinations $\Pi_V(x) \pm \Pi_A(x)$. The combination $V - A$ receives no contribution from perturbation theory and provides a clean probe for chiral symmetry breaking and the quark condensate. $V + A$, on the other hand, allows for a study of perturbative QCD and gluonic operators.

We have compared the two correlators with the predictions of the random instanton liquid and the OPE. The instanton model provides a very accurate description of the $V - A$ correlator for all distances. In the $V + A$ channel the instanton model, supplemented by pQCD corrections with a running coupling constant, also gives an excellent description of the data for $x < 1$ fm.

The remarkable degree of cancellation of nonperturbative effects in the $V + A$ channel provides a unique opportunity to access perturbative corrections well beyond the usual pQCD domain. In the $V - A$ channel, on the other hand, there is an opportunity to extract the dimension $d = 6$, $\langle \bar{q}q \rangle^2$ operator from the data. The result agrees with the SVZ prediction, but the accuracy is limited by the largest invariant mass accessible in $\tau$ decays. In addition to that, the instanton model suggests the presence of a nonsingular $d = 6$ contribution of the same magnitude. Attempts to extract the $d = 4$ gluon condensate operator from the $V + A$ channel fail because in the range of validity of the OPE the $d = 4$ power correction remains a small correction to the pQCD contribution.

We conclude that the range of validity of the OPE in the vector channels is quite small, $x \lesssim 0.3$ fm. This means that there is essentially no "window" in which both the OPE is accurate and the correlation function is dominated by the ground state. Instantons, on the other hand, provide a quantitative tool at all distances. This is true even though the vector channels, because of the smallness of direct instanton effects, are generally considered to be the best system to study the OPE.

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