Noise-Activated Escape from a Sloshing Potential Well

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We treat the noise-activated escape from a one-dimensional potential well of an overdamped particle, to which a periodic force of fixed frequency is applied. Near the well top, the relevant length scales and the boundary layer structure are determined. We show how behavior near the well top generalizes the behavior determined by Kramers, in the case without forcing. Our analysis includes the case when the forcing does not die away in the weak-noise limit. We discuss the relevance of scaling regimes, defined by the relative strengths of the forcing and the noise, to recent optical trap experiments.

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The phenomenon of weak white noise inducing escape from a one-dimensional potential well was studied by Kramers [1]. If ϵ denotes the noise strength (e.g., $\epsilon \propto k_B T$ in thermal systems) and ΔE is the well depth, then the escape rate λ falls off like $\exp(-\Delta E/\epsilon)$ as $\epsilon \to 0$. The case when the trapped particle is overdamped is easiest to analyze. If, after each escape, it is reinjected at the bottom of the well, and a steady state has been set up, then in the interior of the well its position will have a Maxwell-Boltzmann distribution. Kramers showed that this distribution must be modified near the well top, by being multiplied by a "boundary layer function." From the modified distribution, he worked out the weak-noise limit of the escape rate, including its all-important pre-exponential factor, by computing the outgoing flux.

Kramers' formula and its multidimensional generalization have been extended in many ways [2,3]. But a full analysis of escape driven by weak noise, in *periodically modulated* systems, has not yet been performed. Such an analysis would shed light on the Kramers limit of stochastic resonance. It would also clarify the effects of barrier modulation on phase-transition phenomena.

It is now possible to construct a physical system (a mesoscopic dielectric particle that moves, in an overdamped way, within a dual optical trap [4]) that provides a clean experimental test of the three-dimensional Kramers formula. The rate at which thermal noise induces escape agrees well with the predictions of the formula. Adding an external force, of fixed period τ_F, would yield a periodically modulated system [5], of the sort that has not yet been fully analyzed. A complete treatment of the escape of an overdamped particle from a "sloshing potential well" of this sort would surely be desirable.

Smelyanskiy *et al.* treated this phenomenon perturbatively, in one dimension [6]. They derived a Kramers prefactor incorporating f, the periodic forcing strength. It applies if the ratio f/ϵ is set to a constant as $\epsilon \to 0$. That is, the forcing is taken to die away in the weak-noise limit. Lehmann *et al.* [7] treated nonperturbatively the case when f is independent of ϵ, using path integral techniques, and worked out a numerical scheme for computing the f-dependent prefactor. They also examined the "instantaneous escape rate," which in the steady state is a τ_F-periodic function of time. In a simulation of a special case (a well with a perfectly harmonic top), they noted that in the weak-noise limit, the instantaneous escape rate maximum cycles slowly around the interval $[0, \tau_F)$.

In this Letter, we go beyond [6] and [7]. By treating the case $f \propto \epsilon^\alpha$, where α is an arbitrary non-negative power, we determine the relation between their respective scaling regimes. In the weak-noise, weak-forcing limit, there are three physically important length scales near the oscillating well top, of sizes proportional to $\epsilon^{1/2}$, f, and $f^{1/2}$. Crossover behavior will result if $f \propto \epsilon^{1/2}$, and the case $f \propto \epsilon$ can itself be viewed as a crossover regime.

When f is ϵ independent, we use facts on noise-induced transport through unstable limit cycles to illuminate the logarithmically slow "cycling" phenomenon [8]. We compute the instantaneous escape rate as the flux over the oscillating well top, which is an unstable limit cycle. This differs from [6], where the flux through a remote observation point is used. At any t in $[0, \tau_F)$, our normalized escape rate oscillates periodically in $\ln\epsilon$ as $\epsilon \to 0$. We compute the period, and give a physical explanation.

More importantly, we place the case of ϵ-independent periodic forcing firmly in Kramers' framework, by determining how the Maxwell-Boltzmann distribution is modified, in the boundary layer of width $\mathcal{O}(\epsilon^{1/2})$ near the oscillating well top. As $f \to 0$, it approaches the modified distribution of Kramers [1]. The case when $f \propto \epsilon$ in the weak-noise limit is intermediate between the case of ϵ-independent forcing and the case of zero forcing, and its boundary layer structure is intermediate also.

Scaling regimes.—Initially, we work in terms of dimensional quantities. The Langevin equation for a driven Brownian particle in a potential well $U = U(x)$ is

$$m\ddot{x} + \gamma m\dot{x} = -U'(x) + F\nu(t) + \sqrt{2m\gamma k_B T}\,\eta(t). \tag{1}$$

Here γ is the damping, F is a dimensional measure of the driving, ν is a dimensionless periodic function of unit

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amplitude, and η is a standard white noise. In the overdamped (large-γ) limit, the inertial term can be dropped, leaving

$$\dot{x} = -V'(x) + f\nu(t) + \sqrt{\epsilon}\,\eta(t). \quad (2)$$

Here $V = U/\gamma m$, $f = F/\gamma m$, and $\epsilon = 2k_BT/\gamma m$.

Kramers' formula for the $f = 0$ escape rate is

$$\lambda \sim \frac{\omega_s\omega_u}{2\pi\gamma}\exp(-\Delta U/k_BT) = \frac{\sqrt{V''(x_s)\,|V''(x_u)|}}{2\pi}\exp(-\Delta E/\epsilon), \quad (3)$$

where $\omega_s = \sqrt{U''(x_s)/m}$ and $\omega_u = \sqrt{|U''(x_u)|/m}$ are the oscillation frequencies about the bottom x_s and top x_u of the well, and $\Delta E = 2\Delta V$. Equation (3) follows from Kramers' modification of the steady-state Maxwell-Boltzmann weighting $\exp[-U(x)/k_BT]$, i.e., $\exp[-2V(x)/\epsilon]$. If n denotes the inward offset from x_u, his modifying factor is $\mathrm{erfc}[-n/\sqrt{\epsilon/|V''(x_u)|}\,]$.

If $f \neq 0$, there are two regimes, depending on the size of f as $\epsilon \to 0$. Since $[\epsilon] = [\Delta E] = L^2/t$ and $[f] = L/t$, where L denotes length and t denotes time, f will be "large" in the Kramers limit if it is large compared to a quantity with dimensions L/t, namely, $\sqrt{|V''(x_u)|\epsilon}$.

In physical terms, there are two regimes because there are two length scales at the well top. The first is the length scale in Kramers' modification. There is a layer of width $\approx\sqrt{2k_BT/|U''(x_u)|}$, i.e., $\sqrt{\epsilon/|V''(x_u)|}$, within which "physics occurs." This $\mathcal{O}(\epsilon^{1/2})$ quantity is the diffusion length: the distance to within which the particle must approach, to acquire a substantial chance of leaving the well.

If a periodic force is applied, a second length scale becomes important. The well top will oscillate periodically around its unperturbed location x_u by an amount roughly equal to $F/|U''(x_u)|$. If this length scale is substantially larger than the first, escape dynamics should be strongly affected. The crossover occurs when $F \approx \sqrt{2|U''(x_u)|k_BT}$, i.e., when $F \approx \sqrt{2m\omega_u^2 k_BT}$. In normalized units, this criterion is $f \approx \sqrt{|V''(x_u)|\epsilon}$.

So if $f \propto \epsilon^\alpha$ in the Kramers limit ($\epsilon \to 0$), $\alpha > 1/2$ and $0 \leq \alpha < 1/2$ are regimes of weak and strong forcing, respectively. The two regimes should be kept in mind when conducting experiments on escape in periodically driven systems. In the Kramers limit, only when the forcing is much weaker than $\sqrt{2m\omega_u^2k_BT}$ is a simple perturbative modification of Kramers' formula likely to apply.

An illustration would be the room-temperature dual optical trap experiment of McCann *et al.* [4], in which $m \approx 3 \times 10^{-16}$ kg and $\omega_u = (7 \pm 2) \times 10^4\ \mathrm{sec}^{-1}$. The corresponding force magnitude $\sqrt{2m\omega_u^2k_BT}$ is approximately 10^{-13} Newtons. Any repetition of their experiment, with the addition of periodic driving, should take this dividing line into account.

Preliminaries.—Our analysis of the $\alpha = 0$ case, in which f is independent of ϵ, uses *optimal trajectories*. The $\epsilon \to 0$ limit is governed by the action functional

$$\mathcal{W}[t \mapsto x(t)] = \frac{1}{2}\int |\dot{x} + V'(x) - f\nu(t)|^2\,dt. \quad (4)$$

Suppose that $f = 0$. Then the most probable trajectory from x_s to any specified point x' is the one that minimizes $\mathcal{W}[t \mapsto x(t)]$. The minimum is over all trajectories from x_s to x', and all transit times (finite and infinite). There is a single minimizer $t \mapsto x_*(t)$ to each side of x_s, which we term an optimal trajectory. The value $\mathcal{W}[t \mapsto x_*(t)]$, which depends on x' and may be denoted $W(x')$, is the rate at which fluctuations to x' are exponentially suppressed as $\epsilon \to 0$. In the steady state, the probability density ρ of the particle will have the limiting form

$$\rho(x) \sim K(x)\exp[-W(x)/\epsilon], \qquad \epsilon \to 0. \quad (5)$$

The prefactor $K(x)$ must be computed by other means.

Any such $f = 0$ optimal trajectory must satisfy $\dot{x} = +V'(x)$, i.e., be a time-reversed relaxational trajectory. This is due to detailed balance [9]. The trajectory from x_s to x_u is instantonlike: it emerges from x_s at $t = -\infty$ and approaches x_u as $t \to +\infty$. Within the well, $W(x)$ equals $2[V(x) - V(x_s)]$, so $\Delta E \equiv W(x_u)$ equals $2[V(x_u) - V(x_s)]$. Also, K is independent of x.

If $f = 0$, the model defined by the Langevin equation (2) is invariant under time translations. So the optimal trajectory from x_s to x_u is not unique. If $x = x_*(t)$ is a reference optimal trajectory, consider the family

$$t \mapsto x_*^{(\phi)}(t) \equiv x_*\left(t - \frac{\phi}{2\pi}\tau_F\right), \quad (6)$$

where the phase shift ϕ satisfies $0 \leq \phi < 2\pi$, and τ_F is the period of the forcing function $\nu = \nu(t)$. In the Kramers limit of any model with f nonzero but small, the most probable escape trajectory should resemble some trajectory of the form (6). That is, some ϕ_m will be singled out as maximizing the chance of a particle being "sloshed out." A study of the $f \to 0$ limit should yield ϕ_m.

This was the approach of [6]. Suppose that $f \neq 0$. If ΔE is computed by applying (4) to the unperturbed ($f = 0$) optimal trajectory $x = x_*^{(\phi)}(t)$, the first-order [i.e., $\mathcal{O}(f)$] correction to ΔE will be $fw_1(\phi)$, where

$$w_1(\phi) \equiv -\int_{-\infty}^{\infty}\dot{x}_*^{(\phi)}(t)\nu(t)\,dt. \quad (7)$$

It is reasonable to average the Arrhenius factor $\exp(-\Delta E/\epsilon)$ in Kramers' formula over ϕ, from 0 to 2π. If $\langle\cdot\rangle_\phi$ denotes this averaging, then the escape rate will be modified by the driving, to leading order, by a factor $\langle e^{-fw_1(\phi)/\epsilon}\rangle_\phi$. If $\alpha = 1$, i.e., $f = f_1\epsilon$ for some f_1, then Kramers' formula (3) will be altered to

$$\lambda \sim \langle e^{-f_1w_1(\phi)}\rangle_\phi\frac{\sqrt{V''(x_s)\,|V''(x_u)|}}{2\pi}\exp(-\Delta E/\epsilon). \quad (8)$$

Clearly, ϕ_m should be the phase that minimizes $w_1(\phi)$.

Equation (8) is essentially the formula of Smelyanskiy *et al.* [6]. But our derivation makes it clear that their perturbative approach requires that $f \to 0$ rapidly as $\epsilon \to 0$, i.e., that α be sufficiently large. Estimating the minimum of $\mathcal{W}[\cdot]$ by applying it to *unperturbed* optimal trajectories yields a correction to ΔE which is valid only to $\mathcal{O}(f^1)$.

If f is independent of ϵ, then, by Laplace's method,

$$\langle e^{-fw_1(\phi)/\epsilon}\rangle_\phi \sim \frac{1}{\sqrt{2\pi w_1''(\phi_m)f}}\,\epsilon^{1/2}e^{-fw_1(\phi_m)/\epsilon} \tag{9}$$

as $\epsilon \to 0$. This would seemingly suggest that

$$\lambda \sim \frac{\sqrt{V''(x_s)\,|V''(x_u)|}}{2\pi}\,\frac{1}{\sqrt{2\pi w_1''(\phi_m)f}}\,\epsilon^{1/2}\exp(-\Delta E/\epsilon) \tag{10}$$

is the $\alpha = 0$ Kramers formula, with ΔE shifted by $fw_1(\phi_m)$ to leading order. But the prefactor in (10) is correct only in the small-f limit. If $f \propto \epsilon^\alpha$, the $\mathcal{O}(f^1)$ correction to ΔE will be of size $\mathcal{O}(\epsilon^\alpha)$. If $\alpha = 1$, it will alter the prefactor, as in (8). But when $\alpha \le 1/2$, $\mathcal{O}(f^2)$ terms will also affect the prefactor. The most difficult case is $\alpha = 0$, where terms of all orders in f will affect the prefactor. A nonperturbative treatment is called for.

Analysis.—We first remove explicit time dependence, when $f \neq 0$ and τ_F are fixed, by replacing (2) with

$$\dot{x} = -V'(x) + f\nu(y) + \sqrt{\epsilon}\,\eta(t)\,,\qquad \dot{y} = 1\,. \tag{11}$$

Here $0 \le y < \tau_F$, and y is periodic: $y = \tau_F$ is identified with $y = 0$. The state space with coordinates $\mathbf{X} \equiv (x,y)$ is effectively a cylinder. On this cylinder, the oscillating well bottom $x = \tilde{x}_s^{(f)}(t)$ is a stable limit cycle, and the oscillating well top $x = \tilde{x}_u^{(f)}(t)$ is an unstable limit cycle. To stress f dependence, we denote them $\mathbf{X}_s^{(f)}$ and $\mathbf{X}_u^{(f)}$.

To study escape through $\mathbf{X}_u^{(f)}$ as $\epsilon \to 0$, we can use the results of Graham and Tél [10,11]. The $\epsilon \to 0$ limit is governed by a helical, instantonlike optimal trajectory $\mathbf{X} = \mathbf{X}_*^{(f)}(t)$, which spirals out of $\mathbf{X}_s^{(f)}$ and into $\mathbf{X}_u^{(f)}$. It is the most probable escape path in the steady state. ΔE equals $\mathcal{W}[t \mapsto \mathbf{X}_*^{(f)}(t)]$, which must be computed numerically. $\mathbf{X}_*^{(f)}$ would first be computed nonperturbatively, by integrating Euler-Lagrange equations.

As $\mathbf{X}_*^{(f)}$ nears the oscillating well top, it increasingly resembles a time-reversed relaxational trajectory. So, at any specified y, the lth winding of $\mathbf{X}_*^{(f)}$, as it spirals into $\mathbf{X}_u^{(f)}$, has an inward offset n that shrinks geometrically, like ac^{-l}, as $l \to \infty$. Here $a = a(y)$ and c are f dependent, and $c = \exp\{\oint |V''[\tilde{x}_u^{(f)}(t)]|\,dt\}$.

The form (5) for the steady-state probability density generalizes to $K(\mathbf{X})\exp[-W(\mathbf{X})/\epsilon]$. To compute W and K at any specified $\mathbf{X}'$, an optimal trajectory ending at $\mathbf{X}'$ is needed; in general, one different from $\mathbf{X}_*^{(f)}$. An asymptotic analysis of the Smoluchowski equation for the probability density [9,12] shows that W satisfies the Hamilton-Jacobi equation $H(\mathbf{x},\nabla\mathbf{W}) = 0$, where the Hamiltonian $H(\mathbf{x},\mathbf{p})$ equals $\mathbf{p}\cdot\mathbf{D}\cdot\mathbf{p}/2 + \mathbf{u}(\mathbf{x})\cdot\mathbf{p}$. Also, along any trajectory, the density prefactor K satisfies

$$\dot{K} = -(\nabla\cdot\mathbf{u} + D_{ij}\partial_i\partial_j W/2)K\,. \tag{12}$$

Here $\mathbf{u}(x,y) \equiv [-V'(x) + f\nu(y), 1]$ is the drift on the cylinder, and $(D_{ij}) \equiv \mathrm{diag}(1,0)$ is the diffusion tensor. It follows from the Hamilton-Jacobi equation that the Hessian $(\partial_i\partial_j W)$ obeys a Riccati equation along any optimal trajectory [12,13]. This gives a numerical scheme for computing $K(\mathbf{X}')$, starting with an $\mathcal{O}(1)$ value for K on $\mathbf{X}_s^{(f)}$.

In principle, the steady-state escape rate λ can be computed by Kramers' method [1]: evaluating the probability flux through $\mathbf{X}_u^{(f)}$. But this is intricate, due to a subtle problem discovered by Graham and Tél [10,11]. Optimal trajectories that are perturbations of the escape trajectory $t \mapsto \mathbf{X}_*^{(f)}(t)$ intersect one another wildly near $\mathbf{X}_u^{(f)}$. This is because $t \mapsto \mathbf{X}_*^{(f)}(t)$ is a delicate object: a "saddle connection" in the Hamiltonian dynamics sense. In consequence, any $\mathbf{X}'$ near $\mathbf{X}_u^{(f)}$ is reached by an *infinite discrete set* of optimal trajectories, indexed by l, the number of times a trajectory winds around the cylinder before reaching $\mathbf{X}'$. The steady-state density has limiting behavior [12]

$$\rho(\mathbf{X}) \sim \sum_l K^{(l)}(\mathbf{X})\exp[-W^{(l)}(\mathbf{X})/\epsilon], \qquad \epsilon \to 0\,, \tag{13}$$

since W and K are *infinite valued,* not single valued.

It is known [10–12] that, at any fixed y, any $W^{(l)}$ is not quadratic but linear in the offset n from $\mathbf{X}_u^{(f)}$:

$$W^{(l)}(n) \approx \Delta E - |W_{,nn}|[ac^{-l}n - (ac^{-l})^2/2]. \tag{14}$$

$W_{,nn} < 0$ is what, in the absence of multivaluedness, the Hessian matrix element $\partial^2 W/\partial n^2$ would equal at $n = 0$. Along $\mathbf{X}_u^{(f)}$, it obeys the scalar Riccati equation

$$\partial W_{,nn}/\partial y = -W_{,nn}^2 + 2V''[\tilde{x}_u^{(f)}(y)]W_{,nn}\,. \tag{15}$$

$W_{,nn} = W_{,nn}(y)$ is the τ_F-periodic solution of this equation, which is easy to solve numerically. At any y, $W_{,nn}$ equals $2V''(x_u)$ to leading order in f. If V is anharmonic at the well top, deviations from this value will occur.

It is also known [12] that the second term on the right-hand side of (12) tends rapidly to zero along $\mathbf{X}_*^{(f)}$, as it spirals into $\mathbf{X}_u^{(f)}$. So, with each turn, K is multiplied by $\exp[-\oint(\nabla\cdot\mathbf{u})\,dt]$, i.e., by $\exp\{\oint V''[\tilde{x}_u^{(f)}(t)]\,dt\}$. This factor equals c^{-1}. So $K^{(l)} \sim Ac^{-l}$ for some $A = A(y)$. Since $n \sim ac^{-l}$, it follows that, along $\mathbf{X}_*^{(f)}$, $K \sim k_1 n$ as $n \to 0$. Here $k_1 \equiv A/a$, like $W_{,nn}$, is a τ_F-periodic function of y, which quantifies the *linear falloff* of K.

On $[0,\tau_F)$, k_1 turns out to be proportional to $W_{,nn}$ [14]. It can be found by integrating (12) along $\mathbf{X}_*^{(f)}$, as it spirals into $\mathbf{X}_u^{(f)}$. It is the $t \to \infty$ limit of the quotient K/n. Nonconstancy, if any, is due to anharmonicity of V.

Substituting (14) and $K^{(l)} \sim k_1ac^{-l}$ into (13) yields

$$e^{-\Delta E/\epsilon}\sum_{l=-\infty}^{\infty} k_1ac^{-l}\exp\{|W_{,nn}|[ac^{-l}n - (ac^{-l})^2/2]/\epsilon\} \tag{16}$$

as the $\epsilon \to 0$ steady-state probability density ρ, at an inward offset n from the oscillating well top. Summing from $-\infty$ to ∞ is acceptable since the relative errors it introduces are exponentially small, and ignorable. The dependence here on t, i.e., on y, is due to $W_{,nn}$, k_1, and a.

Discussion.—The cycling phenomenon, and much else, follows from the infinite sum (16). To determine its behavior on the $\mathcal{O}(\epsilon^{1/2})$ diffusive length scale near the oscillating well top, set $n = N\epsilon^{1/2}$ with N fixed, and also multiply by $\epsilon^{-1/2}$. (As in the case of no periodic driving, a steady-state density $\tilde{\rho}$ that is normalized to total probability 1 within the well must include an $\epsilon^{-1/2}$ factor.) The resulting expression is invariant under $\epsilon \mapsto c^{-2}\epsilon$. So

$$\tilde{\rho}(n = N\epsilon^{1/2}, t) \sim h_\epsilon^{(f)}(N,t)\exp(-\Delta E/\epsilon), \qquad \epsilon \to 0, \tag{17}$$

where the quantity $h_\epsilon^{(f)}(N,t)$, for any N and any t in $[0,\tau_F)$, is periodic in $\ln\epsilon$ with period $2\ln c$.

In the steady state, the instantaneous escape rate $\lambda(t)$ through the oscillating well top, which equals $(\epsilon/2)(\partial/\partial n)\tilde{\rho}|_{n=0}$, has limiting behavior

$$\lambda(t) \sim (1/2)\epsilon^{1/2}h_\epsilon^{(f)\prime}(0,t)\exp(-\Delta E/\epsilon), \qquad \epsilon \to 0. \tag{18}$$

Thus at any t in $[0,\tau_F)$, the instantaneous escape rate, divided by $\epsilon^{1/2}\exp(-\Delta E/\epsilon)$, ultimately oscillates in $\ln\epsilon$ with period $2\ln c$, i.e., with period $2\{\oint |V''[\tilde{x}_u^{(f)}(t)]|\, dt\}$.

Lehmann *et al.* [7] noted that, on $[0,\tau_F)$, the peak of the function $\lambda(\cdot)$ may shift when ϵ is decreased. Our preceding result indicates this phenomenon is widespread. Its cause is physical. As $\epsilon \to 0$, the most probable trajectory taken by an escaping particle is the helix $t \mapsto \mathbf{X}_*^{(f)}(t)$, along which it moves in a ballistic, noise-driven way. However, once it gets within an $\mathcal{O}(\epsilon^{1/2})$ distance of the oscillating well top, it moves diffusively instead. It is easily checked that such a changeover must take place at a location that cycles slowly around $[0,\tau_F)$, as $\epsilon \to 0$. If $\epsilon \mapsto c^{-2}\epsilon$, the changeover returns to its original location.

If the well top is perfectly harmonic, so that $W_{,nn}$ and k_1 do not depend on t, and also the bottom, it is straightforward to integrate $\lambda(t)$ over $[0,\tau_F)$. We find

$$\lambda \sim \frac{k_1\sqrt{V''(x_s)}}{\sqrt{2\pi}\,\tau_F|V''(x_u)|}\,\epsilon^{1/2}\exp(-\Delta E/\epsilon). \tag{19}$$

It is useful to compare (19) with the perturbative formula (10). They are consistent if k_1 diverges like $f^{-1/2}$ as $f \to 0$. An $f^{-1/2}$ divergence was seen in this special case by Lehmann *et al.* [7], and it occurs widely [14]. It has major consequences. k_1 is the normal derivative of the prefactor K. But K is $\mathcal{O}(1)$ on $\mathbf{X}_s^{(f)}$, and is well behaved in the well interior as $f \to 0$. So there must be a layer of width $\mathcal{O}(f^{1/2})$ near the top of the well, in which K declines linearly to zero. This has been seen numerically [14].

We can now compare the steady-state probability density (17), which is valid on the $\mathcal{O}(\epsilon^{1/2})$ length scale near the oscillating well top, to the density when $f = 0$. The analog of $h_\epsilon^{(f)}(N,t)$, when $f = 0$, is (up to a constant)

$$\operatorname{erfc}[-\sqrt{|V''(x_u)|}\,N] \times \epsilon^{-1/2}\exp(|V''(x_u)|N^2). \tag{20}$$

The erfc is Kramers' boundary function [1], and the exponential is from the Maxwell-Boltzmann distribution.

How can $h_\epsilon^{(f)}(N,t)$, which is defined by an infinite sum, degenerate into such a classical (and t-independent) form when $f \to 0$? The origin of this "weak-driving discontinuity" is clear: the $f \to 0$ limit passes through an intermediate scaling regime, namely, $\alpha = 1$, where the sum (16) is not valid. Light is thrown on this by an integral representation for $\tilde{\rho}(n,t)$ of Smelyanskiy *et al.* [6,15], which is valid if $\alpha = 1$. In our notation, it is (up to a constant)

$$\epsilon^{-1}e^{-2\Delta V/\epsilon}\int_0^\infty e^{[np - p^2/4|V''(x_u)|]/\epsilon}\,e^{-(f/\epsilon)w_1(\varphi)}\,dp, \tag{21}$$

$\varphi = \varphi(p,t) \equiv \phi_m + 2\pi\log[p/2|V''(x_u)|a(t)]/\log c$. The expression (21) is a Maslov-WKB form [16], which we study further elsewhere [14]. It is easy to see that (21) provides an interpolation, across the $\alpha = 1$ regime, from the small-f portion of the $\alpha = 0$ regime to the $f = 0$ case treated by Kramers. When f is nonzero and small, but fixed as $\epsilon \to 0$, applying Laplace's method to (21) yields an infinite sum resembling our sum (16). And when $f = 0$, evaluating (21) yields Kramers' erfc factor.

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