Synchronization of Single-Side Locally Averaged Adaptive Coupling and Its Application to Shock Capturing

G. W. Wei

Department of Computational Science, National University of Singapore, Singapore 117543, Singapore

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We propose a single-sided locally averaged adaptive coupling scheme for the synchronization of spatially extended systems. Coupling and synchronization are analyzed from the viewpoint of image filter construction and numerical dissipation. Single-sided locally averaged coupling is introduced based on the resolution argument of control process. Control sensors are adaptively selected and automatically adjusted according to the magnitude of local oscillations. We demonstrate that the present scheme can effectively suppress and control spatiotemporal oscillations and, thus, provide a powerful approach for shock capturing. Both the Navier-Stokes equation and Burgers' equation are used to illustrate the idea.

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Synchronization phenomenon is of fundamental importance in telecommunication [1], electronic circuits [2], nonlinear optics [3], and chemical and biological systems [4]. The phenomenon has been studied extensively by both numerical and experimental means. It is believed that an in-depth study and understanding of synchronization will greatly benefit the advancement of science and technology. Different types of synchronization, such as identical [1,5], generalized [6], lag, and phase synchronization [7], were proposed. Recently, synchronization and control of spatially extended systems have received great attention [8,9]. For a given system, the degree and rate of synchronization depend vitally on the coupling scheme used. A variety of coupling schemes, such as unidirectional coupling, receptor-product coupling, adaptive coupling, weak coupling, strong coupling, global coupling, and local coupling, have been studied. However, single-sided locally averaged coupling and its effect on the rate and degree of synchronization have not been addressed yet. Moreover, very little is reported on synchronization with respect to the understanding of nonlinear hyperbolic conservation laws, shock capturing, and, in general, computational methodology. The latter has had tremendous impact to science and engineering. In fact, much of the present understanding on synchronization was achieved with the aid of numerical computations.

The main purpose of this Letter is to introduce the synchronization scheme of single-sided locally averaged adaptive coupling and to use it for shock capturing. A nonlinear local gradient based coupling scheme is introduced for spatially extended continuous systems. Nondecreasing functions and nonincreasing functions are designed for oscillation reduction and image edge preservation, respectively. We demonstrate that appropriate coupling of two identical dynamical systems can result in a novel and efficient scheme for shock capturing. The validity and robustness of this scheme are tested by using Burgers' equation and the Navier-Stokes equation.

For simplicity, we consider an identical synchronization, where two coupled systems are exactly of the same type and can be given by a partial differential equation (PDE) of the form

$$u_1 = F(u, u_x, u_{xx}, \ldots) + c(u, w), \tag{1}$$

where $c(u, w)$ is a dissipative coupling term, which is proportional to the difference between the states of two systems, $(u - w)$. Junge and Parlitz [9] proposed an interesting sensor coupling scheme, which utilizes the difference between two localized spatially averaged signals $(\overline{u} - \overline{w})$. Here, $\overline{w}(x, t) = (1/l) \int_{x-l/2}^{x+l/2} w(y, t)\, dy$ is the local average of w over a length l at the position of a *sensor*. The idea behind their coupling scheme is that typical experimental measuring devices have a finite resolution l and measure localized spatial averages of some spatial observable. It is generally true that measuring devices (sensors) and controllers have a finite resolution. However, the system being measured might have an unlimited resolution as it is represented by a continuous PDE. Therefore, we propose a single-sided locally averaged coupling scheme,

$$c(u, w) \propto (u - \overline{w}), \tag{2}$$

where $\overline{w}$ is a localized spatial average of w. It is important to understand that the coupling between two systems given by Eq. (2) is generally designed as a dissipative coupling. However, an interesting observation can be made at the limit of complete identical synchronization [i.e., when there is a strong convergence between the two systems $\|u(x, t) - w(x, t)\| \to 0$ as $t \to \infty$]. From the point of view of image processing, the local average $\overline{w}$ is equivalent to the treatment of w by a low-pass filter. Moreover, at the limit of complete identical synchronization, $(u - \overline{w})$ is equivalent to the treatment of u by a high-pass filter [10]. There is a similar effect on the second system under the same condition. The effect of the single-sided averaged coupling, $(u - \overline{w})$, can also be understood from its semidiscretized form,

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$$u(x_i,t) - \frac{1}{2n+1}\sum_{k=i-n}^{i+n} w(x_k,t), \qquad (n = 0,1,\ldots), \tag{3}$$

where $(2n+1)\Delta x = l$ and Δx is the grid spacing. If $n = 0$, Eq. (3) reduces to the conventional coupling scheme $(u - w)$. For $n = 1$, we have $u(x_i,t) - [w(x_{i-1},t) + w(x_i,t) + w(x_{i+1},t)]/3$, which, at the limit of complete identical synchronization, becomes

$$\lim_{w\to u}\{u(x_i,t) - [w(x_{i-1},t) + w(x_i,t) + w(x_{i+1},t)]/3\} \to -\frac{(\Delta x)^2}{3}\left[\frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{(\Delta x)^2}\right]. \tag{4}$$

Obviously, the term in the last square bracket is a standard finite difference approximation to the second-order derivative operator $(\partial^2 u)/(\partial x^2)$. Such an operator is dissipative in a PDE and is connected to many physical quantities, such as the kinetic energy of a Hamiltonian system. For n larger than 1, a straightforward explanation for Eq. (3) is not available. However, such a case can be studied by numerical experiments.

The proposed single-sided locally averaged coupling expression in Eq. (2) can be used for image processing, pattern recognition, and shock capturing. In these applications, spatially selected treatment is of practical importance. To achieve spatial selectivity, we introduce the following adaptively distributed local sensors:

$$c(u,w) \propto \varepsilon(|u_x|)(u - \overline{w}), \tag{5}$$

where the coupling strength ε is a function of the gradient measurement $|u_x|$. For the purpose of edge-detected pattern recognition, we choose $\varepsilon(|u_x|)$ as a nonincreasing function, e.g., $\varepsilon(|u_x|) = \epsilon\exp[-(|u_x|^2)/(2\sigma^2)]$, where ϵ and σ are constants. For the purpose of noise reduction and oscillation suppression, we choose $\varepsilon(|u_x|)$ as a nondecreasing function, e.g.,

$$\varepsilon(|u_x|) = \epsilon|u_x|^{1/4}, \tag{6}$$

where ϵ is a constant. In fact, an interesting temporal variation adapted coupling scheme was proposed by Boccaletti and Arecchi [11]. The form of the present gradient based function differs from that in Ref. [11] and obviously, many other forms can also be used. For the remainder of the Letter, we restrict ourselves to the application of the present synchronization scheme to shock capturing.

The solution of the inviscid Burgers' equation and the incompressible Navier-Stokes equations at very low viscosity is often difficult to attain due to the possible existence of shock front. Shock wave is a common phenomenon in nature, such as in aerodynamics and hydrodynamics, and is usually described by hyperbolic conservation laws and by inviscid hydrodynamic equations. The construction of numerical schemes that are capable of efficient shock capturing is a challenging task.

To illustrate the present synchronization approach for oscillation reduction, we first consider Burgers' model of turbulence,

$$u_t + uu_x = \nu u_{xx}, \tag{7}$$

where $u(x,t)$ is the dependent variable resembling the flow velocity, and ν characterizes the size of the viscosity. Burgers' equation is an important model for the understanding of physical flows. We consider Eq. (7) using the following initial and boundary conditions:

$$u(x,0) = \sin(\pi x), \qquad u(0,t) = u(1,t) = 0. \tag{8}$$

The fourth-order Runge-Kutta scheme is used for the temporal discretization with a time increment $\Delta t = 0.002$. A discrete singular convolution (DSC) algorithm [12,13] is utilized for spatial discretization with a total of 101 grid points in the computational domain. The DSC algorithm was proposed for computer realization of singular convolutions. The mathematical foundation of the algorithm is the theory of distribution and wavelet analysis. Its use for solving differential equations has been extensively tested [12,13] and further validation is given in Table I. Numerical results of a third-order upwind scheme for convection, in association with a fourth-order central difference scheme for diffusion, is also listed in Table I for a comparison.

Solving Burgers' equation at low viscosity is a challenging task. At $\nu = 0.001$, the numerical solution quickly develops into a sharp shock front near $x = 1$. Severe oscillations occur near the shock front as shown in Fig. 1(a). It should be pointed out that almost all high order numerical schemes exhibit similar oscillations. To eliminate oscillations, we employ the single-sided locally averaged adaptive coupling, Eqs. (5) and (6). Here, $\overline{w}$ is computed by a local three-term average ($n = 1$). Two systems, which are characterized by two viscosities ($\nu_1 = 0.001$ and $\nu_2 = 0.01$), are coupled with a coupling constant of $\epsilon = -80$. It can be seen from Fig. 1(b) that all spurious oscillations are eliminated. However, the synchronized solution is neither the true solution of $\nu = 0.01$ nor that of $\nu = 0.001$. Hence, it is desirable to have an oscillation-free solution at a given low viscosity. To this end, we design an *autosynchronization* approach by choosing two exactly identical systems, i.e., setting $\nu_1 = \nu_2 = 0$. As two exactly identical systems are still coupled, oscillations are suppressed to a certain degree, depending on the coupling constant. For a relatively small coupling constant of $\epsilon = -40$, the solution is oscillatory at early times and become essentially nonoscillatory at a later time [see Fig. 1(c)]. By increasing the coupling constant to $\epsilon = -90$, we have successfully eliminated all spurious

TABLE I. Errors of the DSC and upwind solutions for Burgers' equation ($\nu = 0.01$).

	DSC		Upwind	
t	L_1	L_∞	L_1	L_∞
0.6	4.5(−07)	1.4(−05)	1.3(−04)	3.1(−03)
1.4	1.8(−09)	4.8(−08)	3.2(−05)	5.1(−04)
2.2	6.0(−12)	1.9(−10)	1.1(−05)	1.3(−04)
3.0	2.2(−13)	1.4(−12)	5.2(−06)	4.4(−05)

oscillations as shown in Fig. 1(d). The result of a third-order upwind scheme is also depicted in Fig. 1(d) for a comparison. At the shock front, the upwind scheme cannot completely eliminate oscillations.

To study the size effect of the local averaging and to compare with the upwind scheme further, we consider the inviscid ($\nu = 0$) Burgers' equation [Eq. (7)] with a Riemann-type initial value,

$$u(x,0) = \begin{cases} 1, & \text{if } 0 \le x \le 0.2 \\ 0, & \text{if } 0.2 < x \le 1 . \end{cases} \qquad (9)$$

The spatial and temporal discretizations are the same as in the previous case. The nonlinear coupling [Eq. (6)] is used with $\epsilon = -42$, while the size of the local average in Eq. (3) varies from $n = 1$ to $n = 3$. These results, together with those obtained by using the third-order upwind scheme, are plotted in Fig. 2. The synchronization result obtained with $n = 1$ is perhaps the best available for this problem. It should be noted that the scheme becomes more dissipative as the size of local average is enlarged. As a result, the shock front is more smeared for larger values of n. As in the previous case, the third-order upwind scheme is not as effective as the proposed approach for oscillation suppression.

To further validate the present approach, we consider the two-dimensional Navier-Stokes equation:

$$\mathbf{U}_t + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{U} + \varepsilon(|\nabla \mathbf{U}|)(\mathbf{U} - \overline{\mathbf{W}}), \qquad (10)$$

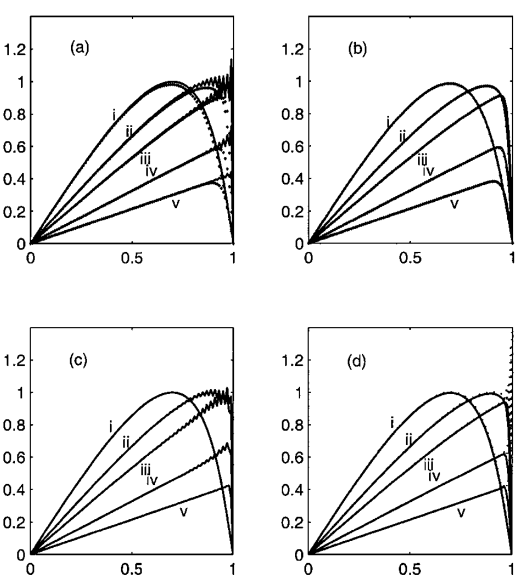

FIG. 1. Synchronization profiles of Burgers' equations at $t = 0.2$ (i), 0.4 (ii), 0.6 (iii), 1.2 (iv), and 2.0 (v). (a) $\epsilon = 0$, solid line: $\nu_1 = 0.001$, diamonds: $\nu_2 = 0.01$; (b) $\epsilon = -80$, solid line: $\nu_1 = 0.001$, diamonds: $\nu_2 = 0.01$; (c) $\epsilon = -40$, $\nu_1 = \nu_2 = 0$; (d) $\epsilon = -90$, $\nu_1 = \nu_2 = 0$. The dots in (d) are obtained by using the upwind scheme.

with the incompressible condition, $\nabla \cdot \mathbf{U} = 0$. Here, $\mathbf{U} = (u, v)^T$ is the velocity vector, $\mathbf{W}$ is the velocity vector of the second system, p is the pressure, and Reynolds number of $\mathrm{Re} = \infty$ defines the Euler equation. The domain of interest is a square $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. Depending on the initial values, this system can be very challenging to solve. For a smooth initial value, the problem is analytically solvable and the validity of the DSC algorithm for this case was tested in Ref. [13].

We now test the present synchronization approach for the Euler equation with sharply varying initial values

$$u(x,y,0) = \begin{cases} \tanh\left(\frac{2y - x}{2\rho}\right), & \text{if } y \le \pi \\ \tanh\left(\frac{3\pi - 2y}{2\rho}\right), & \text{if } y > \pi , \end{cases} \qquad (11)$$
$$v(x,y,0) = \delta \sin(x) .$$

These initial values describe the flow field consisting of horizontal shear layers of finite thickness, perturbed by a small amplitude vertical velocity, making up the boundaries of a jet. This problem is not analytically solvable and is chosen to illustrate the ability of the present approach for providing very fine resolution even on a relatively coarse grid. Pioneering work was done by Bell *et al.* [14] in this field with a second-order Godunov scheme and a projection approach. A state-of-the-art high-order essentially nonoscillatory (ENO) scheme was later constructed by E and Shu [15] to resolve fine vorticity structures.

We consider parameters $\delta = 0.05$ and $\rho = \pi/15$, a case studied by Bell *et al.* [14] with three sets of grids (128^2, 256^2, and 512^2). E and Shu [15] computed this case by using both a spectral collocation code with 512^2 points and their high-order ENO scheme with 64^2 and 128^2 points. The spectral collocation code produced an oscillatory solution at $t = 10$ (see Fig. 1 of Ref. [15]), while the high-order ENO scheme produced a defect at $t = 6$ as the channels connecting the vorticity centers are slightly distorted (see Fig. 2 of Ref. [15]). In the present simulation, we choose a 64^2 grid for the computational domain

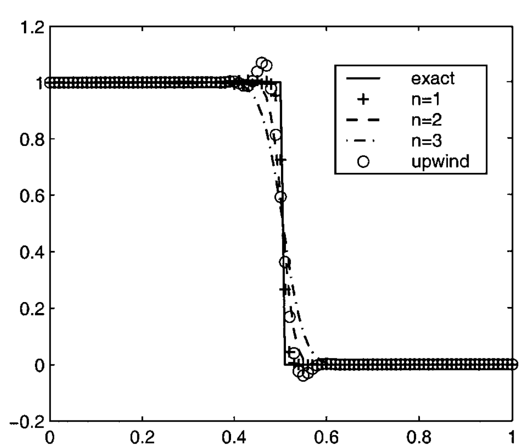

FIG. 2. A comparison of synchronization and upwind approaches for solving the inviscid Burgers' equation ($\nu = 0$, $t = 0.6$) with a Riemann-type initial value.

with a time increment of 0.002. The synchronization prescription given in Eqs. (5) and (6) is used for both velocity components u and v of Eq. (10) with an averaging size parameter of $n = 1$ and a coupling constant of $\epsilon = -80$. The results at different times ($t = 4$, 6, 8, and 10) are plotted in Fig. 3. It is seen that the present solution is smooth and stable for this case. In particular, no distortion is found in vorticity contours at $t = 6$. For early times, present results compare extremely well with those of the spectral collocation code computed with 512^2 points. There are no spurious numerical oscillations during the entire process.

In conclusion, we propose the approach of synchronization as a robust, reliable, and practical algorithm for shock wave computations. A single-sided locally averaged coupling scheme is introduced based on the resolution argument of control sensors. The coupling strength in spatially extended systems is adaptively varied according to the magnitude of the local gradient of the system. The resulting coupled systems are analyzed from the viewpoint of image filters and numerical dissipation. The size effect of the local averaging is studied. The proposed algorithm is validated by using Burgers' equation and the incompressible Navier-Stokes equation. A high accuracy discrete singular convolution algorithm [12,13] is utilized for the numerical simulation and results are compared with those obtained by using an upwind scheme.

For Burgers' equation, computational accuracy and reliability is tested. At a very low viscosity, Burgers' equation develops spurious oscillations, which can be eliminated by coupling to another Burgers' equation with a higher viscosity. To make the algorithm practical for shock capturing at any given viscosity, two truly identical systems are coupled. It is found that the oscillations can be completely suppressed above a minimum coupling strength.

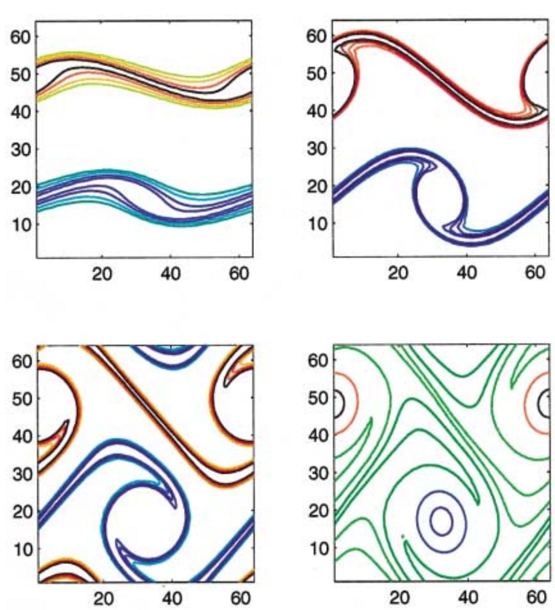

FIG. 3 (color). The vorticity contours of the synchronization solution of the 2D Euler equation. Upper left: $t = 4$; upper right: $t = 6$; lower left: $t = 8$; lower right: $t = 10$.

The scheme becomes more dissipative as the size of the local average is enlarged, as indicated by solving the inviscid Burgers' equation with a Riemann-type initial value. The proposed algorithm is found to be more effective than a third-order upwind scheme for oscillation reduction.

To further validate the present approach for shock capturing, the Navier-Stokes equation is considered. The simulation of this system is an acid test for ordinary methods. The present results are better than those of the ENO scheme [15]. This indicates that the proposed approach has a great potential for being used as a practical algorithm for the simulation of fluid flows and computational physics in general.

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