Unitarity Bounds and the Cuspy Halo Problem

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Conventional cold dark matter cosmological models predict cuspy halos which are in apparent conflict with observations. We show that unitarity arguments imply interesting constraints on two proposals to address this problem: collisional dark matter and strongly annihilating dark matter. Efficient scattering in both implies $m \lesssim 12$ GeV and $m \lesssim 25$ GeV, respectively. We also show that the strong annihilation in the second scenario implies the presence of elastic scattering. Recent evidence suggests a collisional scenario where the cross section scales inversely with velocity—we argue superelastic processes are likely involved. Exceptions and implications for searches are discussed.

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I. Introduction.—There is a long history of efforts to constrain dark matter properties from galactic structure (e.g., [1]). Recent numerical simulations [2] sharpen the predictions of cold dark matter (CDM) structure formation models, and apparent discrepancies with the observed properties of structures from galactic to cluster scales are uncovered. The main one that has attracted a lot of attention is the cuspy halo problem, namely, that CDM models predict halos that have a high density core or have an inner profile that is too steep compared to observations ([3], but see also [4]). This has encouraged several proposals that dark matter might have properties different from those of conventional CDM (see [5]and summary therein).

On the other hand, general principles of quantum mechanics impose nontrivial constraints on some of these models. We focus here on the proposals of collisional or strongly self-interacting dark matter (SIDM) by [6] and of strongly annihilating dark matter (SADM) by [7]. Both require a high level of interaction by particle physics standards: an elastic scattering cross section of $\sigma_{\rm el} \sim 10^{-24}(m_X/\text{GeV})$ cm^2 for the former and an annihilation cross section of $\sigma_{\rm ann} v_{\rm rel} \sim 10^{-28}(m_X/\text{GeV})$ cm^2 for the latter, where m_X is the particle mass, and $v_{\rm rel}$ is the relative velocity of approach. The proposed dark matter is therefore quite different from the usual candidates such as the axion or neutralino. We show that the unitarity of the scattering matrix, together with a few reasonable assumptions, imposes interesting particle mass bounds as well as other physical constraints. This is done while making minimal assumptions about the nature of the interactions. Our results complement constraints from experiments or astrophysical considerations, e.g., [8].

Griest and Kamionkowski [9] previously derived similar mass bounds related to the freeze-out density of thermal relics, assuming two-body final states. In section II, we provide a general derivation for arbitrary final states using the classic optical theorem [10]. We summarize our findings in section III, discuss exceptions to our bounds, and discuss other solutions to the cuspy halo problem.

II. Deriving the unitarity bounds.—Different versions of the unitarity bounds can be found in many textbooks, and can be most easily understood using nonrelativistic quantum mechanics (e.g., [11]), which is probably adequate for our purpose. However, the results and derivation given here might be of wider interest, e.g., for estimating thermal relic density. Here we follow closely the field theory treatment of [12].

The optical theorem [10] is a powerful consequence of the unitarity of the scattering matrix S, i.e., $S^\dagger S = 1$, which implies $(1 - S)^\dagger(1 - S) = (1 - S^\dagger) + (1 - S)$, or

$$\int d\gamma \, \langle\beta|1 - S|\gamma\rangle\langle\gamma|1 - S^\dagger|\alpha\rangle = 2\,\text{Re}\langle\beta|1 - S|\alpha\rangle, \tag{1}$$

where α and β represent two specified states and γ represents a complete set of states with measure $d\gamma$. By using the definition of the scattering amplitude $A_{\beta\alpha}$,

$$\langle\beta|1 - S|\alpha\rangle \equiv -i(2\pi)^4\delta^4(p_\beta - p_\alpha)A_{\beta\alpha}\,, \tag{2}$$

where p_β and p_α are the total four-momenta, one obtains

$$\int d\gamma\,(2\pi)^4\delta^4(p_\alpha - p_\gamma)\,|A_{\gamma\alpha}|^2 = 2\,\text{Im}A_{\alpha\alpha} \tag{3}$$

if $\beta = \alpha$ in Eq. (1). We are interested in the case where α represents a two-body state of $X + X$ or $X + \bar{X}$ approaching each other. The final state γ, on the other hand, is completely general, and the integration over γ covers the entire spectrum of possible final states. To be more precise, suppose $|\alpha\rangle = |k_1, s_1^z; k_2, s_2^z; n\rangle$ where s_1^z and s_2^z represent the spin states of the two incoming particles with spins s_1 and s_2 (in our particular case, $s_1 = s_2$), while k_1 and k_2 are their respective four-momenta, and n labels the particle types (e.g., mass, etc.). Recalling that $d\sigma/d\gamma \propto |A_{\gamma\alpha}|^2$, Eq. (3) gives, in the center-of-mass (c.m.) frame (adopted hereafter, i.e., $\mathbf{k}_1 + \mathbf{k}_2 = 0$),

$$\int d\gamma \,\frac{d\sigma}{d\gamma}\,(\alpha \to \gamma) = \frac{\text{Im}A_{\alpha\alpha}}{2(E_1 + E_2)\,|\mathbf{k}_1|}\,, \tag{4}$$

where the left-hand side is exactly the total cross section. This is the optical theorem. It states that the total cross section for scattering from a two-body initial state *to all possible* final states equals the imaginary part of the two-body *to two-body* forward scattering amplitude.

To use this theorem, we expand the scattering amplitude in terms of partial waves, i.e., states labeled as $|\mathbf{k}_{\rm tot}, E_{\rm tot}, j, j^z, \ell, s, n\rangle$, where $\mathbf{k}_{\rm tot}$ is the total linear momentum (= 0 in the center of mass frame), $E_{\rm tot}$ is the total energy, j is the total angular momentum, j^z is its z component, ℓ is the orbital momentum, and s is the total spin. By inserting appropriate complete sets of partial wave outer products into Eq. (2), we obtain

$$A_{\beta\alpha} = 4i(2\pi)^2[E_1' + E_2'/|\mathbf{k}_1'|]^{1/2}[E_1 + E_2/|\mathbf{k}_1|]^{1/2}\sum_{j,j^z}\sum_{\ell',s',\ell,s}\langle\ell's'n'|1 - S|\ell sn\rangle_{j,E_{\rm tot}} \times \sum_{\ell^{z\prime},s^{z\prime}}\langle s's^{z\prime}\,|\,s_1^{z\prime}s_2^{z\prime}\rangle_{s_1',s_2'}\langle jj^z\,|\,\ell^{z\prime}s^{z\prime}\rangle_{\ell',s'}\langle\hat{\mathbf{k}}_1'\,|\,\ell'\ell^{z\prime}\rangle\sum_{\ell^z,s^z}\langle ss^z\,|\,s_1^zs_2^z\rangle^*_{s_1,s_2}\langle jj^z\,|\,\ell^zs^z\rangle^*_{\ell,s}\langle\hat{\mathbf{k}}_1\,|\,\ell\ell^z\rangle^*, \tag{5}$$

where the crucial assumption is that S is rotationally invariant and so j and j_z are conserved, in addition to energy conserving. The notation $\langle\ell's'n'|1 - S|\ell sn\rangle_{j,E_{\rm tot}}$ emphasizes that S is diagonal in j, j^z, and $E_{\rm tot}$ but the j^z dependence drops out because S commutes with $J_x \pm iJ_y$. The inner products $\langle ss^z\,|\,s_1^zs_2^z\rangle_{s_1,s_2}$ and $\langle jj^z\,|\,\ell^zs^z\rangle_{\ell,s}$ give the Clebsch-Gordon coefficients, and $\langle\hat{\mathbf{k}}_1\,|\,\ell\ell^z\rangle = Y_{\ell\ell^z}(\hat{\mathbf{k}}_1)$ is the spherical harmonic function. We assume $\hat{\mathbf{k}}_1 = \hat{\mathbf{z}}$, in which case $Y_{\ell\ell^z}(\hat{\mathbf{k}}_1) = \delta_{\ell^z,0}\sqrt{2\ell + 1/(4\pi)}$. The index β denotes a two-body final state $|k_1', s_1'; k_2', s_2'; n'\rangle$.

By setting $\beta = \alpha$, and averaging over the spin states (i.e., $(2s_1 + 1)^{-1}(2s_2 + 1)^{-1}\sum_{s_1^z,s_2^z}$) on both sides of Eq. (4), the optical theorem, we obtain [12]

$$\sigma_{\rm tot} = \frac{2\pi}{|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)}\sum_j(2j + 1) \times \sum_{\ell,s}\mathrm{Re}\langle\ell sn|1 - S|\ell sn\rangle_{j,E_{\rm tot}}. \tag{6}$$

This gives the total spin-averaged cross section for scattering from $X + X$ or $X + \bar{X}$ to all possible final states.

For $X + \bar{X}$ annihilation, we exclude from the above the contribution due to elastic scattering (where the type and mass of particles do not change, i.e., $X + \bar{X} \to X + \bar{X}$, implying $|\mathbf{k}_1'| = |\mathbf{k}_1|$) [9]. To do so, we need the following expression for two-body to two-body scattering cross section:

$$\frac{d\sigma}{d\beta}\,d\beta = \frac{|A_{\beta\alpha}|^2}{4(E_1 + E_2)\,|\mathbf{k}_1|}\,(2\pi)^4\delta^4(p_\beta - p_\alpha)d\beta, \tag{7}$$

We average over initial spin states and integrate over outgoing momenta, but focus on the elastic contribution (n' in $|\beta\rangle = |k_1', s_1'; k_2', s_2'; n'\rangle$ is set to n in $|\alpha\rangle$) [12]:

$$\sigma_{\rm el} = \frac{\pi}{|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)}\sum_j(2j + 1) \times \sum_{\ell,s,\ell',s'}|\langle\ell's'n|1 - S|\ell sn\rangle_{j,E_{\rm tot}}|^2. \tag{8}$$

The above is the total cross section for elastic scattering (note, the same expression also describes $X + X \to X + X$ elastic scattering) that has to be subtracted from $\sigma_{\rm tot}$ to yield the total inelastic scattering cross section, which is relevant for annihilation into all possible final states:

$$\sigma_{\rm inel} = \frac{\pi}{\mathbf{k}_1^2(2s_1 + 1)(2s_2 + 1)}\sum_j(2j + 1) \times \sum_{\ell,s}\Bigg[1 - |\langle\ell sn|S|\ell sn\rangle|^2 - \sum_{\ell'\neq\ell,s'\neq s}|\langle\ell's'n|1 - S|\ell sn\rangle|^2\Bigg]. \tag{9}$$

From Eqs. (6) and (9), we can derive two bounds,

$$\sigma_{\rm tot} \leq 4\pi[|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)]^{-1}\sum_j\sum_{\ell,s}2j + 1, \tag{10}$$

$$\sigma_{\rm inel} \leq \pi[|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)]^{-1}\sum_j\sum_{\ell,s}2j + 1. \tag{11}$$

The first inequality uses $|\langle\ell sn|S|\ell sn\rangle|^2 \leq 1$, obtained from $\int d\gamma\,\langle\ell sn|S^\dagger|\gamma\rangle\langle\gamma|S|\ell sn\rangle \geq |\langle\ell sn|S|\ell sn\rangle|^2$ and $S^\dagger S = 1$. A similar bound can be derived for $\sigma_{\rm el}$ as well, which coincides exactly with that for $\sigma_{\rm tot}$.

We pause to note that the above bounds assume only unitarity and the conservation of total energy and linear and angular momentum. No assumptions are made about the nature of the particles, whether they are composite or pointlike. Nor do we assume the number of particles in the final states. To obtain useful limits from the bounds, we take the low velocity limit. Assuming the scattering amplitude $A_{\beta\alpha}$ is an analytic function of $\mathbf{k}_1$ as $\mathbf{k}_1 \to 0$ (exceptions will be discussed in section III), and noting that $k^\ell\langle\hat{\mathbf{k}}|\ell\ell^z\rangle$ is a polynomial function of $\mathbf{k}$, we expect the ℓ partial wave contribution to $A_{\beta\alpha}$ [Eq. (5)] to scale as $|\mathbf{k}_1|^\ell$. This means that, in the low velocity limit, as is relevant for our purpose (typical velocity dispersion in halos range from 10 to 1000 km/s $\ll c$), the $\ell = 0$ or s-wave contribution dominates. Setting $\ell = 0$ in Eqs. (10) and (11):

$$\sigma_{\rm tot} \leq 16\pi/(m_Xv_{\rm rel})^2, \qquad \sigma_{\rm inel}v_{\rm rel} \leq 4\pi/(m_X^2v_{\rm rel}), \tag{12}$$

where $|\mathbf{k}_1|^2 = k_1^2 = m_X^2|\mathbf{v}_2 - \mathbf{v}_1|^2/4 = m_X^2v_{\rm rel}^2/4$ is used. The second inequality agrees with [9]. Hence,

$$\sigma_{\rm tot} \lesssim 1.76 \times 10^{-17}\ {\rm cm}^2 \left[\frac{\rm GeV}{m_X}\right]^2 \left[\frac{10\ {\rm km\,s^{-1}}}{v_{\rm rel}}\right]^2, \quad (13)$$

$$\sigma_{\rm inel} v_{\rm rel} \lesssim 1.5 \times 10^{-22}\ {\rm cm}^2 \left[\frac{\rm GeV}{m_X}\right]^2 \left[\frac{10\ {\rm km\,s^{-1}}}{v_{\rm rel}}\right]. \quad (14)$$

Furthermore, if $\sigma_{\rm inel}$ is bounded from below, say $\sigma_{\rm inel} \geq \sigma_{\rm ann}$, one can derive a lower bound on $\sigma_{\rm el}$ using Eqs. (8) and (9), and setting $\ell = 0$. Defining $\langle X \rangle_J \equiv [(2s_1 + 1)(2s_2 + 1)]^{-1} \sum_{j,\ell,s} (2j + 1)X$, using S at the moment to denote $\langle \ell sn | S | \ell sn \rangle$, and noting that $\langle 1 \rangle_J = 1$ for $\ell = 0$, it can be shown that $(\pi/k_1^2)(1 - \langle |S| \rangle_J^2) \geq (\pi/k_1^2)(1 - \langle |S|^2 \rangle_J) \geq \sigma_{\rm inel} \geq \sigma_{\rm ann}$, which implies $\langle |S| \rangle_J \leq \sqrt{1 - k_1^2 \sigma_{\rm ann}/\pi}$. Also, $\sigma_{\rm el} \geq (\pi/k_1^2)\langle |1 - S|^2 \rangle_J \geq (\pi/k_1^2)\langle (1 - |S|)^2 \rangle_J \geq (\pi/k_1^2)(1 - \langle |S| \rangle_J)^2$. Combining, we have

$$\sigma_{\rm el} \geq (\pi/k_1^2)[1 - \sqrt{1 - k_1^2 \sigma_{\rm ann}/\pi}]^2. \quad (15)$$

This tells us that the elastic scattering cross section cannot be arbitrarily small given a nonvanishing inelastic cross section, e.g., via annihilation.

The above three bounds are the main results of this section. Two more results will be useful for our later discussions. For two-body to two-body processes, recall that the ℓ, ℓ' contribution to $A_{\beta\alpha}$ scales as $|\mathbf{k}_1|^{\ell} |\mathbf{k}_1'|^{\ell'}$. By using $d\sigma/d\Omega = |A_{\beta\alpha}|^2 (|\mathbf{k}_1'|/|\mathbf{k}_1|)/[64\pi^2(E_1 + E_2)^2]$ [obtained from Eq. (7) by integrating over β, except for solid angle Ω], it can be seen that for elastic scattering, where $|\mathbf{k}_1'| = |\mathbf{k}_1|$,

$$d\sigma/d\Omega \rightarrow {\rm const}[1 + O(v_{\rm rel})] \quad (16)$$

as $|\mathbf{k}_1| \rightarrow 0$. For inelastic scattering where the system gains kinetic energy by losing rest mass (e.g., deexcitation of a composite particle or annihilation), since $|\mathbf{k}_1'|$ approaches a nonzero value as $|\mathbf{k}_1| \rightarrow 0$, we have

$$d\sigma/d\Omega \rightarrow ({\rm const}/v_{\rm rel})[1 + O(v_{\rm rel})] \quad (17)$$

instead in the low velocity limit. The opposite case where the particle gains mass is discussed in [12].

III. Discussion.—We can derive the following four constraints for strongly self-interacting dark matter [6] and strongly annihilating dark matter [7].

(i) The range $\sigma_{\rm el} \sim 10^{-24}$–$10^{-23}\ {\rm cm}^2(m_X/{\rm GeV})$ is given by [5] for SIDM to yield the desired halo properties. Using the lower $\sigma_{\rm el}$, and $v_{\rm rel} \sim 1000$ km/s as appropriate for clusters, we obtain from Eq. (13) $m_X \lesssim 12$ GeV for collisional dark matter.

(ii) The annihilation cross section from [7], $\sigma_{\rm ann} \times v_{\rm rel} \sim 10^{-28}\ {\rm cm}^2(m_X/{\rm GeV})$, together with Eq. (14) and $v_{\rm rel} \sim 1000$ km/s, gives us a bound of $m_X \lesssim 25$ GeV for strongly annihilating dark matter.

(iii) For SADM, efficient annihilation (a form of inelastic scattering) inevitably implies some elastic scattering as well. From Eq. (15), and using $v_{\rm vel} \sim 1000$ km/s as before, we have

$$\sigma_{\rm el} \gtrsim 4 \times 10^{-22}\ {\rm cm}^2[{\rm GeV}/m_X]^2 \times [1 - \sqrt{1 - 7 \times 10^{-5}(m_X/{\rm GeV})^3}\,]^2. \quad (18)$$

There are two simple limiting cases: when m_X is close to the upper bound of 25 GeV, $\sigma_{\rm el} \gtrsim 4 \times 10^{-22}$ cm^2; when m_X is small, $\sigma_{\rm el} \gtrsim 5 \times 10^{-31}\ {\rm cm}^2(m_X/{\rm GeV})^4$. Hence, elastic scattering is inevitable in this scenario, but can be reduced by having a sufficiently small mass.

(iv) Recent simulations suggest that the simplest version of SIDM might fail to match simultaneously the observed halo properties from dwarf galaxies to clusters [5,13] (see also [14]), which have $v_{\rm rel}$ ranging over 3 orders of magnitude. It was suggested that an *elastic* scattering cross section of $\sigma \propto 1/v_{\rm rel}$ might solve the problem. But, as we have argued in Eq. (16), elastic scattering generally implies $\sigma \rightarrow$ const in the small velocity limit. Hence, $\sigma \propto 1/v_{\rm rel}$ likely requires *inelastic* processes. As Eq. (17) shows, processes in which the kinetic energy increases ($|\mathbf{k}_1'| > |\mathbf{k}_1|$ in c.m. frame) can give such a velocity dependence. SADM provides a concrete example. More generally, the net kinetic energy increase (superelasticity) must be taken into account when considering the viability of a model with $\sigma \propto 1/v_{\rm rel}$, e.g., it may delay core collapse and make the core larger. Note, however, that the general considerations in the last section do not forbid an elastic cross section that increases as $v_{\rm rel}$ decreases, e.g., the $O(v_{\rm rel})$ term in Eq. (16) can have a negative coefficient. A $1/v_{\rm rel}$ power law might be used to approximate such a cross section, but likely only for a limited range of $v_{\rm rel}$. An example is the neutron-neutron scattering cross section, which approaches a constant for $|\mathbf{k}_1| \lesssim 10^{-2}$ GeV, and scales roughly as $1/v_{\rm rel}$ only for $10^{-2} \lesssim |\mathbf{k}_1| \lesssim 5 \times 10^{-2}$ GeV [15].

It is helpful to mention here the possible exceptions to the above limits. Our bounds are obtained from Eqs. (13) and (14), which are the $\ell = 0$ (s-wave) versions of Eqs. (10) and (11). The argument for putting $\ell = 0$ in the small velocity limit assumes the analyticity of $A_{\beta\alpha}$ at $\mathbf{k}_1 = 0$. The latter breaks down if the interaction is long ranged, e.g., Coulomb scattering. This is unlikely to be relevant, because there are strong constraints on dark matter with such long-range interaction [16]. Our argument for the dominance of s-wave scattering can also be invalid if there is a resonance. However, given that the scattering cross section should vary smoothly over 3 orders of magnitude in velocities from dwarfs to clusters, a resonance seems unlikely. Finally, the most likely situation in which the bounds break down is if the particle has a large enough size, or the interaction has a large enough effective range, R, such that $|\mathbf{k}_1|R > 1$ (e.g., see [17]). In such cases, higher partial waves in addition to s waves, generally contribute, and $\sigma_{\rm tot} \lesssim 64\pi R^2$ and our arguments turn into a limit on R [9]. The condition $|\mathbf{k}_1|R > 1$ gives the most stringent constraint on R for $v_{\rm rel} = 10$ km/s, as appropriate for dwarf galaxies:

$R \gtrsim 10^{-9}$ cm(GeV$/m_X$). One can compare this with R for neutron-neutron scattering $\sim 10^{-13}$ cm [15].

It is intriguing that the halo structure might be telling us the elementary properties, in particular the mass, of dark matter. It is interesting that several proposals to address the cuspy halo problem, such as warm dark matter [18] and fuzzy dark matter [19], make explicit assumptions about the mass of the particles—$m_X \sim 1$ keV and $m_X \sim 10^{-22}$ eV, respectively. For SIDM and SADM, astrophysical considerations generally put constraints only on the cross section per unit mass. We have shown here that unitarity arguments imply a rather modest mass for both scenarios as well. It is also worth pointing out that our arguments, with suitable modification to take into account Bose enhancement and multiple incoming particles, can be extended to cover dark matter in the form of a Bose condensate, as has been proposed as yet another solution to the cuspy halo problem [20]. They generally require small masses as well, $\lesssim$10 eV.

A few issues are worth further investigation. Wandelt *et al.* [8] recently argued that a version of SIDM, where the dark matter also interacts strongly with baryons, is experimentally viable, but requires $m_X \gtrsim 10^5$ GeV, or $m_X \lesssim 0.5$ GeV. Our bound here is inconsistent with the large mass region (but see exceptions above); experimental constraints on the low mass region will be very interesting ($\sigma_{\rm el} \lesssim 10^{-25}$ cm^2). It would be useful to find a micro-physics realization of the collisional scenario or its variant, where σ scales appropriately with velocity to match observations. The impact of inelastic collisions on halo structures is worth exploring in more detail. It is also timely to reconsider possible astrophysical solutions to the cuspy halo problem, such as the use of mass loss mechanisms [21]. We hope to examine some of these issues in the future.

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[1] S. Tremaine and J. E. Gunn, Phys. Rev. Lett. **42**, 407 (1979).

[2] See, e.g., J. F. Navarro, C. S. Frenk, and S. D. M. White, Astrophys. J. **490**, 493 (1997); A. V. Kravtsov, A. A. Klypin, J. S. Bullock, and J. R. Primack, Astrophys. J. **502**, 48 (1998); B. Moore, T. Quinn, F. Governato, J. Stadel, and G. Lake, Mon. Not. R. Astron. Soc. **310**, 1147 (1999); Y. P. Jing and Y. Suto, Astrophys. J. Lett. **529**, 69 (2000).

[3] See, e.g., R. Flores and J. Primack, Astrophys. J. **427**, L1 (1994); B. Moore, Nature (London), **370**, 629 (1994).

[4] F. C. Van Den Bosch, B. E. Robertson, and J. J. Dalcanton, Astron. J. **119**, 1579 (2000).

[5] R. Davé, D. N. Spergel, P. J. Steinhardt, B. Wandelt, astro-ph/0006218, 2000 (to be published).

[6] D. N. Spergel and P. J. Steinhardt, Phys. Rev. Lett. **84**, 3760 (2000).

[7] M. Kaplinghat, L. Knox, and M. S. Turner, astro-ph/0005210 2000 (to be published).

[8] See, e.g., B. Wandelt, R. Davé, G. R. Farrar, P. C. McGuire, D. N. Spergel, and P. J. Steinhardt, astro-ph/0006344, 2000; J. Miralda-Escude, astro-ph/0002050, 2000.

[9] K. Griest and M. Kamionkowski, Phys. Rev. Lett. **64**, 615 (1990).

[10] E. Feenberg, Phys. Rev. **40**, 40 (1932); N. Bohr, R. E. Peierls, and G. Placzek, Nature (London) **144**, 200 (1939).

[11] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977).

[12] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK, 1995), Vol. 1, Sec. 3.6–3.7.

[13] N. Yoshida, V. Springel, S. D. M. White, and G. Tormen, Astrophys. J. Lett. **544**, L87 (2000); C. Firmani, E. D'Onghia, and G. Chinarini, astro-ph/0010497, 2000 (to be published).

[14] C. Firmani, E. D'Onghia, V. Avila-Reese, G. Chincarini and X. Hernández, Mon. Not. R. Astron. Soc. **315L**, 29 (2000); J. A. Sellwood, Astrophys. J. Lett. **540**, 1 (2000); A. Burkert, Astrophys. J. Lett. **534**, 143 (2000); C. S. Kochanek and M. White, Astrophys. J. **543**, 514 (2000).

[15] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952).

[16] A. Gould, B. T. Draine, and R. Romani, Phys. Lett. B **238**, 337 (1990).

[17] A. Kusenko and P. J. Steinhardt (to be published).

[18] C. J. Hogan and J. J. Dalcanton, Phys. Rev. D **62**, 063511 (2000); see also V. K. Narayanan, D. N. Spergel, R. Davé, and C. P. Ma, astro-ph/0005095, 2000.

[19] W. Hu, R. Barkana, and A. Gruzinov, astro-ph/0003365 2000.

[20] P. J. E. Peebles, astro-ph/0002495, 2000; J. Goodman, New Astron. **5**, 103 (2000); I. Tkachev and A. Riotto, Phys. Lett. B **484**, 177 (2000).

[21] J. F. Navarro, V. R. Eke, and C. S. Frenk, Mon. Not. R. Astron. Soc. **283**, L72 (1996).