Prandtl-Number Dependence of Heat Transport in Turbulent Rayleigh-Bénard Convection

Guenter Ahlers and Xiaochao Xu

Department of Physics and iQUEST, University of California, Santa Barbara, California 93106
(Received 13 November 2000)

We present measurements of the Nusselt number $\mathcal{N}$ as a function of the Rayleigh number R and the Prandtl number σ in cylindrical cells with aspect ratios $\Gamma = 0.5$ and 1.0. We used acetone, methanol, ethanol, and 2-propanol with Prandtl numbers $\sigma = 4.0$, 6.5, 14.2, and 34.1, respectively, in the range $3 \times 10^7 \lesssim R \lesssim 10^{11}$. At constant R, $\mathcal{N}(R, \sigma)$ varies with σ by only about 2%. This result disagrees with the extrapolation of the Grossmann and Lohse theory beyond its range of validity, which implies a decrease by 20% over our σ range, but agrees with their recent extension of the theory to small Reynolds numbers.

During the last decade or two a great deal of effort has been devoted to the study of turbulent convection of fluids heated from below [1–4]. One of the central issues has been the global heat transport of the system, as expressed by the Nusselt number

$$\mathcal{N} \equiv \lambda_{\text{eff}}/\lambda. \tag{1}$$

Here

$$\lambda_{\text{eff}} \equiv qd/\Delta T \tag{2}$$

is the effective thermal conductivity of the convecting fluid (q is the heat-current density, d the height of the sample, and ΔT the imposed temperature difference), and λ is the conductivity of the quiescent fluid. For convection between parallel horizontal plates heated from below the Nusselt number should depend only on the Rayleigh number

$$R \equiv \alpha g d^3 \Delta T/\kappa \nu, \tag{3}$$

the Prandtl number

$$\sigma \equiv \kappa/\nu, \tag{4}$$

and the sample geometry. Here α is the thermal expansion coefficient, g is the gravitational acceleration, κ is the thermal diffusivity, and ν is the kinematic viscosity. We will consider cylindrical samples, characterized by their aspect ratio $\Gamma \equiv D/d$ where D is the diameter.

For $R \gtrsim 10^7$ it was shown in recent experiments [5,6] that $\mathcal{N}(R, \sigma)$ could not be represented by a single power law $\mathcal{N} = \mathcal{N}_0 R^\gamma$ as had often been assumed or predicted [2,7,8], but that the recent prediction of Grossmann and Lohse (GL) [4] based on the decomposition of the kinetic and the thermal dissipation into boundary layer and bulk contributions gives an excellent fit to the R dependence of the data for the particular value $\sigma = 4.0$. For $\sigma \gtrsim 2$ the GL prediction is given by

$$\mathcal{N} = a\sigma^{-1/12}R^{1/4} + b\sigma^{-1/7}R^{3/7}, \tag{5}$$

where a and b are to be determined by fitting to experimental data for a particular Γ. As one can see, this function gives not only the dependence of $\mathcal{N}$ on R, but contains also a specific dependence on σ. Once a and b are determined from a set of measurements at one particular Γ and σ, the σ dependence is given uniquely by the theory without the opportunity for further adjustment.

Here we present new measurements of $\mathcal{N}(R, \sigma, \Gamma)$ over the range $3 \times 10^7 \lesssim R \lesssim 10^{11}$ and $4 \lesssim \sigma \lesssim 34$ [9]. The original theory [4], if extrapolated by using Eq. (5) beyond its proper range of validity, gives approximately a 20% decrease of $\mathcal{N}$ at constant R and Γ over this σ range. Contrary to this, our data show that $\mathcal{N}$ decreases by only about 1% or 2% as σ increases. Motivated by some of these results [9], GL recently extended their theory to smaller Reynolds numbers of the large-scale flow which prevails in turbulent RBC [10].

The apparatus used by us was described previously [5]. Here we report measurements for two cells with $\Gamma = 0.50$ and 1.00. They had diameters $D = 8.70$ cm and heights $d = 17.4$ and 8.70 cm, respectively. The mean temperature of the sample was held nearly constant at 32 °C. We used the fluids acetone, methanol, ethanol, and 2-propanol. On the basis of published values of their properties [11] they have $\sigma = 3.97$, 6.48, 14.2, and 34.1, respectively. Thus our measurements cover almost one decade of σ. Elsewhere [6] it was shown that corrections based on a nonlinear temperature profile for the heat current entering (leaving) the bottom (top) of the sidewall must be applied. We based these corrections on model 2 of Ref. [6].

Since σ varies over a wide range, the published fluid properties [11] are accurate enough for its estimate from Eq. (4). However, the Nusselt number at constant R is expected to change only very little with σ, and thus values of $\mathcal{N}$ and R with no significant variation of the systematic errors from one fluid to another must be obtained from Eqs. (1)–(3), from the experimentally determined heat currents and temperature differences, and from the fluid properties. It turns out that the published properties do not meet this very stringent requirement, and new measurements had to be made.

In a separate set of experiments we determined λ below the onset of convection and the critical temperature difference ΔT_c at onset for all the fluids. For this purpose a thin cell had to be used so that ΔT_c, corresponding
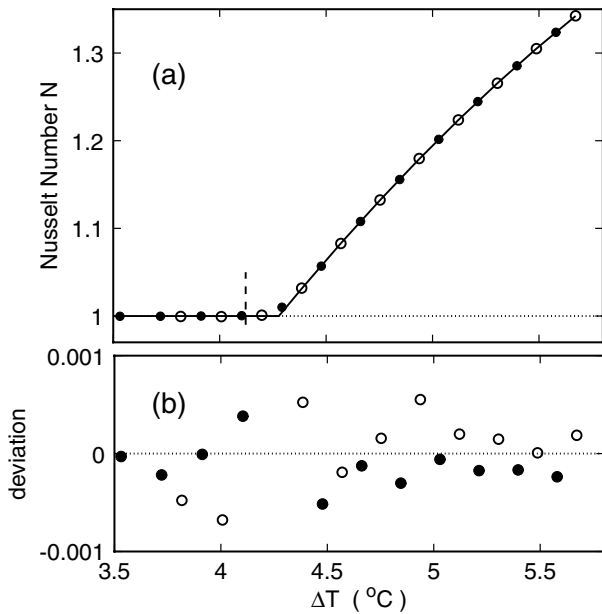
FIG. 1.   The Nusselt number for ethanol and a cell with $d =$ 0.1540 cm as a function of the temperature difference across the fluid. The vertical dashed line is the estimated value $\Delta T_{c,\text{est}}$. The lower figure shows the deviations from the fit of Eqs. (6)–(8) to the data. Open circles: increasing ΔT. Solid circles: decreasing ΔT.

to $R = R_c = 1708$, was of the order of a few °C [see Eq. (3)]. We used a cell with $d = 0.1540 \pm 0.0005$ cm and $D = 8.94 \pm 0.05$ cm ($\Gamma = 58$). It could be evacuated for measurements of the parallel wall conductance, and it could be filled with various fluids without disassembly. As a check on our procedures, we also made measurements with toluene and water which have exceptionally well-known properties [12]. As an example, we show in Fig. 1 results in the form of the Nusselt number $\mathcal{N}$ as a function of ΔT for ethanol. There is a sharp onset of convection at $\Delta T_c = 4.28$ °C. The value $\Delta T_{c,\text{est}} = 4.12$ obtained with the properties given in Ref. [11] is shown by the dashed line. It is a few % too small.

In order to determine best values for λ and ΔT_c, we fitted the equations

$$\mathcal{N} \equiv \lambda_{\text{eff}}/\lambda = 1 + S_1\epsilon + S_2\epsilon^2, \qquad \epsilon > 0, \quad (6)$$

$$\mathcal{N} = 1, \qquad \epsilon < 0, \qquad (7)$$

with

$$\epsilon \equiv \Delta T/\Delta T_c - 1 \qquad (8)$$

to the data, least-squares adjusting λ, ΔT_c, S_1, and S_2. For ethanol this fit is shown by the solid line in Fig. 1a, and deviations from it are given in Fig. 1b (the two points closest to onset were omitted). For comparison, we evaluated the literature values of λ, and of ΔT_c for our d. All the results are given in Table I.

The value of λ is used directly in the calculation of $\mathcal{N}$ for the turbulence experiments in much thicker cells.

TABLE I.   The thermal conductivity λ in W/m °C and the critical temperature difference ΔT_c in °C for a cell with $d = 0.1540$ cm. The uncertainty of λ (ΔT_c) due to the scatter in the measurements generally is less than ± 0.0001 W/m °C (± 0.002 °C). The fourth and fifth columns give the corresponding values estimated from literature data for the fluid properties. The last column gives the initial slope S_1 of $\mathcal{N}$.

Fluid	λ	ΔT_c	λ_{est}	$\Delta T_{c,\text{est}}$	S_1
Acetone	0.1545	1.036	0.1593	1.085	1.11
Methanol	0.1961	2.361	0.1979	2.524	1.26
Ethanol	0.1631	4.278	0.1672	4.122	1.26
2-propanol	0.1334	5.851	0.1334	5.189	1.26
Toluene	0.1310	2.220	0.1314	2.295	1.28
Water	0.6187	16.11	0.6191	16.62	1.24

The values of ΔT_c, together with R_c, d, g, and Eq. (3), yield the property combinations $\alpha/\kappa\nu$. These in turn can be used to evaluate Rayleigh numbers for the turbulence experiments in the much thicker cells.

Results for turbulent RBC are shown on the customary logarithmic scales in Fig. 2. There we show only the data for acetone and $\Gamma = 0.5$ (circles) and for 2-propanol and $\Gamma = 1.0$ (squares). The other data were omitted to avoid overcrowding, but on the scale of this figure they fall nearly on a common straight line through all the results. Thus, within the resolution of this figure there is no noticeable Γ or σ dependence for $0.5 \leq \Gamma \leq 1.0$ and $4 \leq \sigma \leq 34$. Also shown are the results of Niemela et al. (crosses); they tend to be slightly below our data, as would be expected [2,13] for the much smaller $\sigma \simeq 0.7$.

In order to display the results with much greater resolution, the reduced Nusselt number $\mathcal{N}R^{-2/7}$ is shown as a function of $\log_{10}(R)$ in Fig. 3 for $\Gamma = 0.5$ and in Fig. 4 for
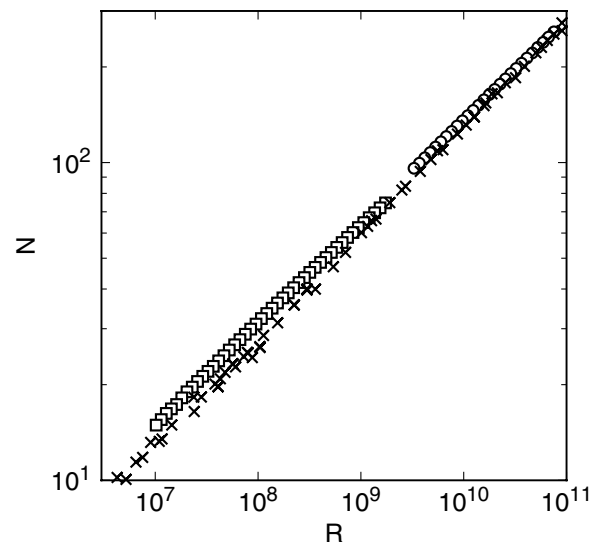

FIG. 2.   The Nusselt number as a function of R on logarithmic scales for acetone ($\sigma = 3.94$) and $\Gamma = 0.5$ (open circles), and for 2-propanol ($\sigma = 34.1$) and $\Gamma = 1.0$ (open squares). The crosses are the data of Ref. [3]. A correction for the wall conduction based on model 2 of Ref. [6] was applied to all data.
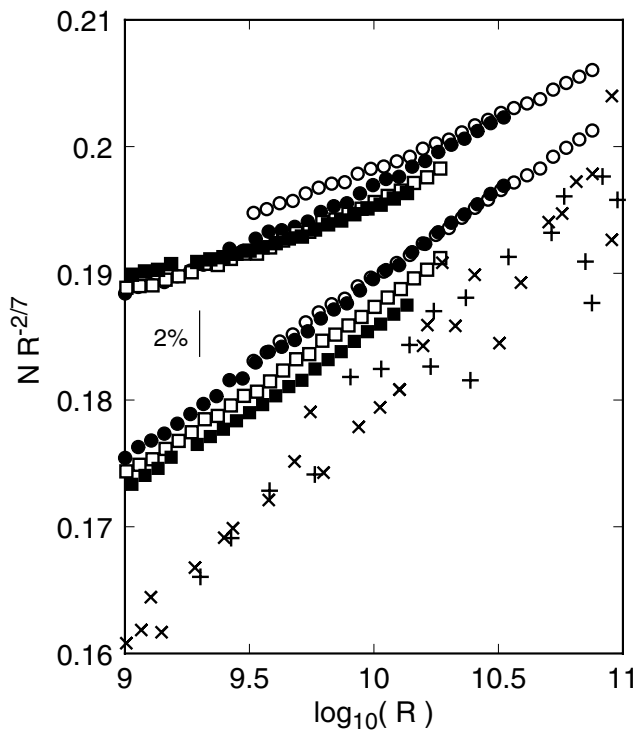
FIG. 3.   The reduced Nusselt number $\mathcal{N} R^{-2/7}$ as a function of the logarithm of the Rayleigh number $\log_{10}(R)$ for the cell with aspect ratio $\Gamma = 0.5$. Open circles: acetone. Solid circles: methanol. Open squares: ethanol. Solid squares: 2-propanol. For the upper set of data the wall correction to the heat current was neglected, and for the lower one it was included based on model 2 of Ref. [6]. The crosses and pluses correspond to the data of Refs. [3] and [14], respectively, after a wall correction based on model 2 of Ref. [6] was applied.
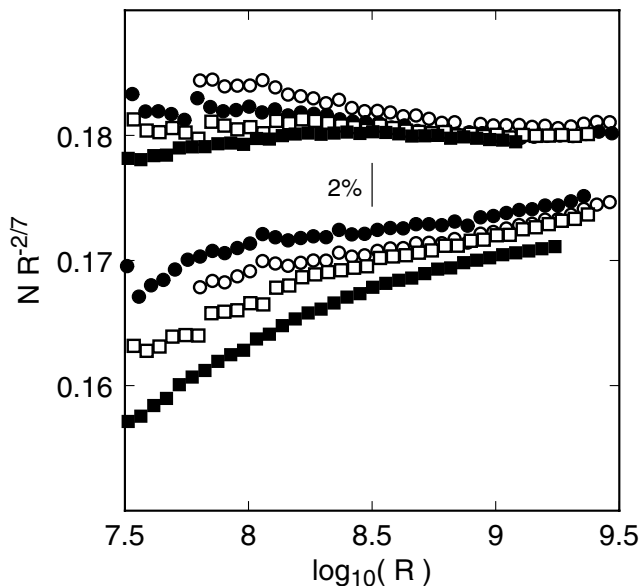

FIG. 4.   The reduced Nusselt number $\mathcal{N} R^{-2/7}$ as a function of the logarithm of the Rayleigh number $\log_{10}(R)$ for the cell with aspect ratio $\Gamma = 1.0$. The symbols are as in Fig. 3.

$\Gamma = 1.0$. Here we do not mean to attach any special significance to the exponent $2/7$, and use it only for historical reasons and to facilitate comparison of our high-resolution graphs with the previous literature. For each cell the data cover about two decades of R, and collectively they span the range of R from 3×10^7 to 10^{11}. In these figures, the top data sets were obtained by neglecting the correction for the heat current passing through the sidewall. The lower sets were obtained after a correction for the wall current based on a nonlinear temperature profile calculated numerically using model 2 of Ref. [6]. Also shown in Fig. 3 are the data of Refs. [3] (crosses) and [14] (pluses) which are for $\sigma \simeq 0.7$ and $\Gamma = 0.5$. Consistent with expectations for the smaller σ [2,13], these data fall slightly below ours. Interestingly, they were slightly higher than our results and agreed less well with each other before the nonlinear wall correction was applied [5].

In Fig. 5 we show $\mathcal{N} R^{-2/7}$ as a function of σ for several values of R. The top part is for $\Gamma = 0.5$ and the bottom is for $\Gamma = 1.0$. In Fig. 5a we also show (as open symbols) approximate values estimated from the results of
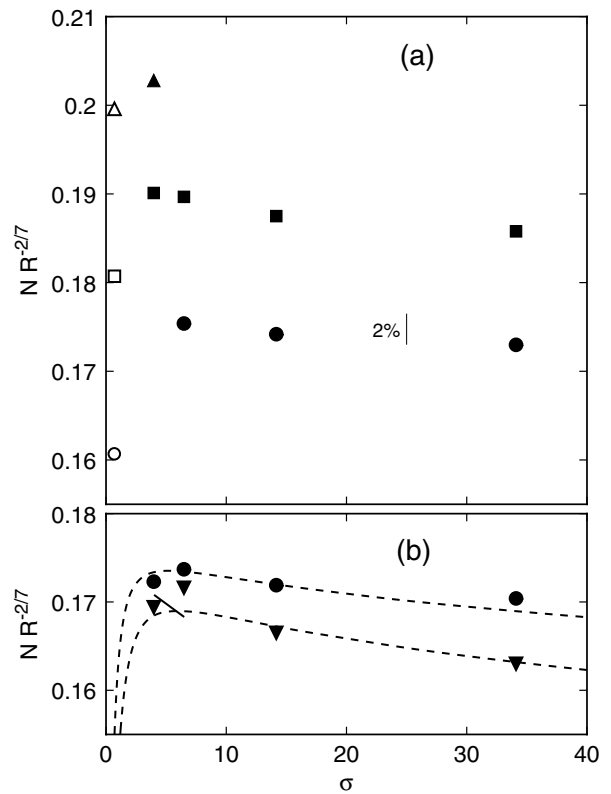

FIG. 5.   The reduced Nusselt number $\mathcal{N}/R^{2/7}$ as a function of the Prandtl number σ. (a) $\Gamma = 0.5$. (b) $\Gamma = 1.0$. Up pointing triangles: $R = 10^{11}$. Squares: $R = 10^{10}$. Circles: $R = 10^9$. Down pointing triangles: $R = 10^8$. Solid symbols: this work. Open symbols: estimates based on the data of Refs. [3] and [14] (see Fig. 3) after the wall correction discussed in Ref. [6] (model 2) was applied. Short solid line in (b): results of Ref. [15]. The dashed lines in (b) are the fit of the theory of Ref. [10] to the data.
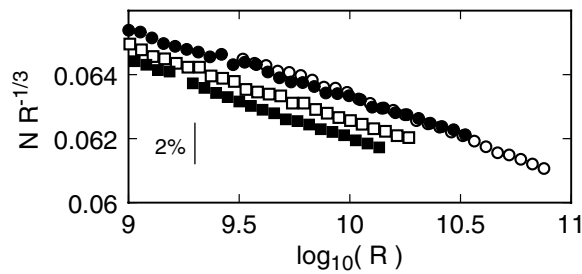
FIG. 6. The reduced Nusselt number $\mathcal{N} R^{-1/3}$ as a function of the logarithm of the Rayleigh number $\log_{10}(R)$ for the cell with aspect ratio $\Gamma = 0.5$. The symbols are as in Fig. 3. Only the results after the wall correction based on model 2 of Ref. [6] are shown.

Refs. [3] and [14] as displayed in Fig. 3. At fixed R one sees that $\mathcal{N}$ varies by only about 2% or 3% over our range of σ. For $\sigma \gtrsim 6$ there is a slight trend for $\mathcal{N}$ to decrease with σ, but it is difficult to be sure that this decrease exceeds remaining systematic errors. The acetone data for $\Gamma = 1$, as well as a comparison of the data from Refs. [3] and [14] with ours for $\Gamma = 0.5$, suggest that $\mathcal{N}(\sigma)$ has a maximum near $\sigma \simeq 6$. The observed very weak σ dependence is inconsistent with an extrapolation beyond its range of validity of the original theory [4] of GL which predicted a decrease of $\mathcal{N}$ by about 20% as σ increased from 4 to 34 [see Eq. (5)]. However, it agrees quite well with the very recent prediction of GL [10] which is based on an extension of the original theory into a parameter range of relatively small Reynolds numbers of the large-scale circulation. For $\Gamma = 1$ the parameters of the extended GL theory were evaluated [10] from a fit to the experimental data, and the results for $R = 10^8$ and 10^9 are shown in Fig. 5b by the dashed lines.

In Fig. 5b we also compare our results with those of Liu and Ecke (LE) [15] for R near 10^8 which are represented by the short solid line. LE made their measurements in a cell with a square cross section with a height-to-length ratio of 1.29. Thus we would expect only approximate agreement with our $\Gamma = 1.0$ data. Nonetheless, our data agree well with them both in their magnitude and in their σ dependence. Our measurements are inconsistent with the experimental results of Ashkenazi and Steinberg who report a much larger decrease of $\mathcal{N}$ with σ [16].

Finally, we examine the extent to which the present results for $\mathcal{N}(R, \sigma)$ are consistent also with some earlier predictions. There are numerous papers which cannot all be cited [13]. Here we focus only on a prediction made four decades ago by Kraichnan [17] on the basis of a mixing-length model (see also Ref. [2]). For large R and $\sigma \gtrsim 1$, a σ-independent $\mathcal{N}$ at constant R was predicted. Our data are compatible with that prediction since we can-

not rule out that the very small decrease of $\mathcal{N}$ with increasing σ is due to systematic errors. However, the model also predicts that $\mathcal{N} \propto R^{1/3}$ at constant σ. Such a dependence should yield a constant value for $\mathcal{N} R^{-1/3}$ in Fig. 6. However, the data show a mild R dependence which would be compatible with an effective exponent of about 0.318 in the R range of the figure, but incompatible with an exponent of $1/3$.

[1] Recent summaries of experimental and theoretical work can be found in Refs. [2–4].

[2] E. D. Siggia, Annu. Rev. Fluid Mech. **26**, 137 (1994).

[3] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, Nature (London) **404**, 837 (2000).

[4] S. Grossmann and D. Lohse, J. Fluid Mech. **407**, 27 (2000).

[5] X. Xu, K. M. S. Bajaj, and G. Ahlers, Phys. Rev. Lett. **84**, 4357 (2000).

[6] G. Ahlers, Phys. Rev. E **63**, 015303 (2001).

[7] B. Castaing, G. Gunaratne, F. Heslot, L. Kadanoff, A. Libchaber, S. Thomae, X. Z. Wu, S. Zaleski, and G. Zanetti, J. Fluid Mech. **204**, 1 (1989).

[8] B. I. Shraiman and E. D. Siggia, Phys. Rev. A **42**, 3650 (1990).

[9] Some of our results were presented before, in Proceedings of the Institute for Theoretical Physics Mini-Workshop on High Rayleigh Number Convection, 2000 (http://online.itp.ucsb.edu/online/hydrot00conv/ahlers/).

[10] S. Grossmann and D. Lohse, preceding Letter, Phys. Rev. Lett. **86**, 3316 (2001).

[11] T. E. Daubert and R. P. Danner, *Physical and Thermodynamic Properties of Pure Chemicals* (Hemisphere Publishing Corporation, New York, 1989).

[12] M. L. V. Ramires, C. A. Nieto de Castro, Y. Nagasaka, A. Nagashima, R. A. Perkins, and W. A. Wakeham, J. Phys. Chem. Ref. Data **29**, 133 (2000), and references therein.

[13] Interesting recent results as well as references to the earlier literature may be found, for instance, in S. Cioni, S. Ciliberto, and J. Sommeria, J. Fluid Mech. **335**, 111 (1997); R. M. Kerr and J. R. Herring, J. Fluid Mech. **419**, 325 (2000).

[14] X. Chavanne, F. Chilla, B. Chabaud, B. Castaing, J. Chaussy, and B. Hébral, J. Low Temp. Phys. **104**, 109 (1996).

[15] Y. Liu and R. E. Ecke, Phys. Rev. Lett. **79**, 2257 (1997).

[16] S. Ashkenazi and V. Steinberg, Phys. Rev. Lett. **83**, 3641 (1999).

[17] R. Kraichnan, Phys. Fluids **5**, 1374 (1962).