Green's Function Probe of a Static Granular Piling

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We present an experiment which aims to investigate the mechanical properties of a static granular assembly. The piling is a horizontal 3D granular layer confined in a box. We apply a localized extra force at the surface and the spatial distribution of stresses at the bottom is obtained (the mechanical Green's function). For different types of granular media, we observe a linear pressure response whose profile shows one peak centered at the vertical of the point of application. The peak's width increases linearly with increasing depth. This Green's function seems to be in at least qualitative agreement with predictions of elastic theory.

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Understanding the exact mechanical status of static or quasistatic granular assemblies is still an open and debated issue [1,2]. So far, there is no consensus on how to express correctly the stress distributions in a granular piling under various boundary conditions. Traditional approaches of soil mechanics typically use elastoplastic modeling for granular materials [3] and constitutive relations are obtained empirically from standard triaxial tests. In this picture, for small deformations, an elasticlike behavior is assumed and a set of elliptic partial differential equations (PDE) is then used to get the stress/strain distributions. For larger strains, the Coulomb plasticity theory is adapted to model granular flows, which involves hyperbolic (propagative) PDE's in regions experiencing yield [4]. At the granular scale, several recent experiments and simulations have evidenced the presence of a rather large distribution of contact forces [5] between the grains as well as force chains [6,7] spanning a volume of 10 to 15 grain sizes. Hence, it is clear that a rigorous passage from a microscopic to a macroscopic mechanical description that would include this mesoscopic disorder is an arduous task and a challenging problem of statistical physics. Consequently, these studies have triggered alternative theoretical approaches. One of them is based on a scalar stochastic modeling [8] for the contact force redistributions. In the large scale limit, this vision provides a diffusivelike picture (parabolic equation) for the stress transmission properties. Another approach incorporates the vectorial and propagative character of the contact forces between the grains [9–11]. This picture, when extended to the continuum limit, predicts simple relations between the components of the stress tensor and implies hyperbolic (i.e., propagative) PDE's for the stress fields [9,12]. A recent framework bridges the two last approaches [13]. From a fundamental point of view, there is no reason why a hyperbolic equation would be expected in general besides in the special case of isostatic (i.e., minimally connected) packing as it was initially suggested by Edwards [14] and rigorously derived by Moukarzel [15] and Tkachenko *et al.* [16]. But this applies essentially to frictionless and infinitely rigid grains and nothing is known in general for granular assemblies with friction. Several reproducible experiments were recently performed on a sand heap (see [17], and references therein) and on a granular column [18,19]. It was shown that a hyperbolic modeling (for example, the oriented stress linearity model [12]) is indeed able to reproduce some of the observed phenomenology. But it is worth noting that the parameters entering in these hyperbolic models are so far phenomenological constants, calculated *a posteriori* and therefore, do not provide us with really predictive statements. Furthermore, the agreement between the available experiments and the hyperbolic models does not rule out the pertinence of elliptic models [20,21].

Here we present an experiment probing the response of static granular assemblies to a local stress perturbation (a Green's function). This is probably the most basic experiment allowing a precise discrimination between the different approaches and which should reveal the real mechanical nature of static granular assemblies [2]. We operate on a horizontal 3D granular assembly confined in a box and the spatial distribution of stresses at the bottom is monitored which provides an important piece of evidence in order to inform this currently debated issue.

Already at the most basic level, measuring meaningful stresses in granular assemblies is a nontrivial question. Generally, experimentalists measure stresses by calibration of devices that are deformed or displaced as a result of a local force distribution on the probe surface. They are confronted to three fundamental problems: (i) the response of the probe depends on the history of the preparation as evidenced on the sand heap [17] (which might be a physically relevant issue), (ii) the probe surface has a limited number of contacts with the granular medium which is at the origin of an inherent fluctuations scale (whose importance should decrease when the probe size increases), and (iii) the physical characteristics of the probe itself may have an influence on the measurements.

In this last situation, large deformations of the device may change drastically the local force distribution creating arching effects around the probe. The stress probe we

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use here is made of a thin metallic membrane of thickness $e = 100\ \mu$m welded on a cylinder. The deformation of the membrane (less than 1 μm) is monitored using a sensitive capacitive technique. To avoid the formation of a vault around the probe and break the history dependence when building the sand layer, the pile is slightly vibrated after each height increase and the sand is randomly poured by hand layer by layer. This procedure produces some fluctuations from one step to the other but in the average, we checked that, locally, the hydrostatic pressure relation $P = \rho g h$ is well recovered (with a precision of 5%, which is the typical uncertainty on the packing fraction determination). For all the experiments we describe here, we use this procedure to prepare the sand pile.

Probing the response of a granular static piling to a localized perturbation is *a priori* a difficult issue. It is of common experience that it is quite easy to drill a hole in a sandy surface when pushing it weakly with the tip of a finger. Therefore, since the perturbing stress must not create plastic reorganization of the grains, only weak perturbations must be applied. Consequently, the detection of the response signal is likely to be rapidly hidden within the noise when depth increases. The typical pressure we apply at the surface of the pile is achieved by a piston P (see Fig. 1) of mass $M = 5g$ with a surface $A = 1$ cm^2 (5.10^2 Pa). In order to increase the signal/noise ratio, we use a lock-in detection technique. More precisely, a local stress modulation is achieved using a periodic magnetic field created by an electric current in a coil surrounding the piston (see C in Fig. 1). In the piston, a permanent magnet is inserted and the coil current is driven by a low

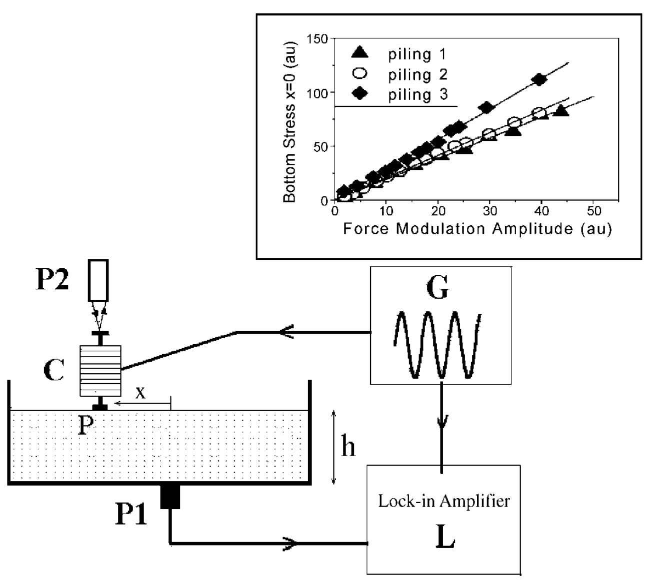

FIG. 1. Sketch of the experimental display: G, low frequency generator; C, electrical coil; P, piston; $P1$, stress probe; $P2$, displacement probe; L, lock-in amplifier (see text for a detailed description). Inset: Test of the response linearity for three independent experiments on the same probe $P2$. The axes (stress versus applied force) are labeled in arbitrary units.

frequency generator (G). Therefore, the modulation of the magnetic field in the coil creates a force modulation on the piston. The signal of the stress probe $P1$ at the bottom of the piling is then directed to the lock-in amplifier L synchronized by the generator exciting the source. Note that a sensitive displacement probe $P2$ monitors the piston position to check that no plastic yield occurs during the data collection. The relative horizontal position x between the piston and the probe $P1$ is varied. We operate in the low frequency limit such that we are basically probing the static properties of the granular piling. The applied extra modulated force is driven at $f = \omega/2\pi = 80$ Hz and we checked that the exact choice of this frequency modulation does not matter in this limit (between 10 and 120 Hz). We also verified that within a reasonable time scale (several tenths of minutes), we do not observe slow variation of the response. Because of the finite sizes of the piston and of the probe, the signal hence obtained is the convolution of the mechanical response function (the Green's function) by the width of the source and the width of the probe. We measured the intrinsic experimental width $W_0 = 10$ mm and we found the convolution effects to be negligible as soon as $h > 3$ cm (corresponding to 3 times the probe diameter).

Here, we report experiments on their granular media with different size d and shape. We use $d \simeq 1$ mm "aquarium sand" and $d \simeq 300\ \mu$m "Fontainebleau sand." The grain shape is rather rough and the size polydispersity is around 50%. We also use monodisperse glass beads (diameter $d = 1.5$ mm). We tested that, in the limit where no "sinking" of the piston inside the pile is observed, the response is linear in the value of the imposed stress. We also found that the value of the slope relating the applied force to the observed stress may depend strongly on granular configurations around the probe. This "realization dependence" causes difficulties to calibrate precisely the probe at 80 Hz on a granular column. This is evidenced in the inset of Fig. 1 where we display the stress measurement for three different piles obtained in the same conditions. The response amplitude after detection by the lock-in amplifier at $x = 0$ is plotted as a function of the force modulation amplitude F. Note that the force values stem from a calibration on a static water column and not on the granular pile response at 80 Hz. The frequency of the signal is small enough such that we have no significant phase shift between the force and the detected stress. Moving the point of application of the force, we change the horizontal distance x between the piston and the probe and plot the pressure profile $\sigma_{zz}(x)$ for a given depth h of sand.

The response function $P(x)$ shows one single peak centered at the vertical of the point of application of the force; we did not observe the two separated bumps as predicted by hyperbolic models [2,10], even when increasing the depth up to 10 cm which corresponds to the limit of our detection possibilities. In Fig. 2a, we display the response functions rescaled by an amount P^* so that they present the same

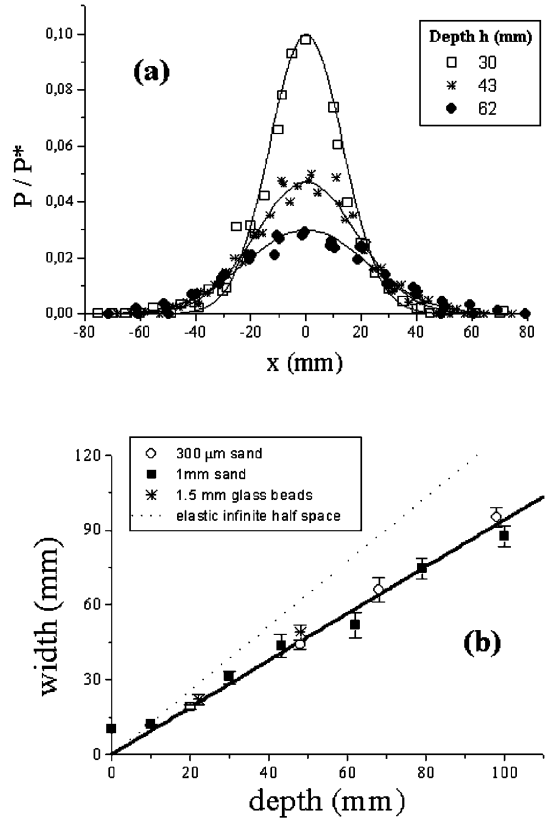

FIG. 2. Horizontal stress distribution in response to a localized solicitation (Green's function). (a) Green's function $P(x) = \sigma_{zz}(x)/P^*$ measured at three different depths for $d = 1$ mm sand. See text for definition of the rescaling factor P^*. (b) Half amplitude width W of the response function as a function of depth h for three different granular materials (see legend). The straight line is the best linear fit: $W = 0.94h$.

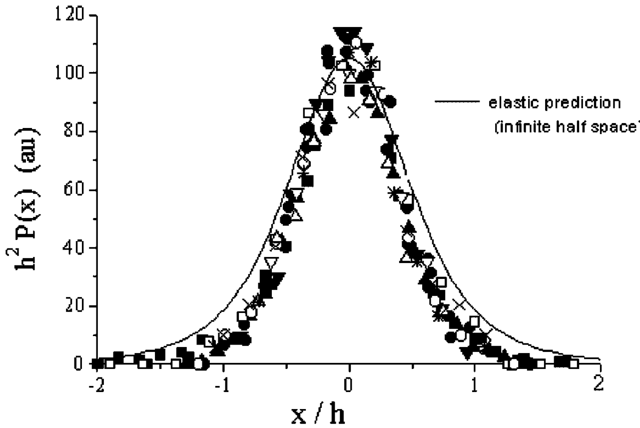

FIG. 3. Rescaled Green's function $h^2P(x)$ as a function of the rescaled horizontal axis x/h, for different depths and different types of granular materials. Aquarium sand: $d = 300$ μm [$h = 30$ mm (■), 62 mm (●), 79 mm (▲), 100 mm (▼)]; Fontainebleau sand: $d = 1$ mm [$h = 19$ mm (□), 48 mm (○), 68 mm (△), 97 mm (▽)]; glass beads: $d = 1$ mm [$h = 28$ mm (*), 48 mm (×)]. The straight line is the theoretical response of an elastic infinite half-space.

area under the curve (i.e., constant applied force). Importantly, we checked that for three different probes situated at three difference places on the bottom plate, this procedure provides us with the same stress distribution. In Fig. 2a, we display three response functions $P(x)$ obtained at different depths h for the $d = 1$ mm sand. We find that for all granular media studied, the width at half amplitude w increases linearly when increasing the depth (whenever $h > W_0$). The slope is independent of the material used (see Fig. 2b). This is in qualitative agreement with the prediction of elastic theory. Here we provide a comparison with the only known exact solution in 3D obtained for an infinite half-space as computed by Boussinesq and Cerruti last century [22]:

$$\sigma_{zz} = \frac{-3F}{2\pi} \frac{z^3}{(x^2 + z^2)^{5/2}},$$

where F is the applied force. In Fig. 3 we plot the response functions at different depths h such that the response $P(x)$ is rescaled by z^2 and the horizontal axis is $x^* = x/h$. We see that for all the granular material tested, the curves are collapsing onto the same function. The response is clearly sharper than the elastic Lorentzian prediction obtained for a semi-infinite medium. We get $w/h = 0.94 \pm 0.05$ instead of $w/h = 1.13\ldots$ (i.e., 20% sharper). One problem here is to account correctly for the bottom boundary condition which is not a trivial issue since many choices can be done and it requires an adapted elastic code. Nevertheless, we are aware of an analytical calculation performed on the same problem in 2D and in 3D by Claudin [23]. Rough and smooth boundary conditions are used. Both calculations show a clear sharpening of the response of about 20% in 2D and 6% in 3D when compared to the exact semi-infinite solution in 2D. Thus the issue is subtle and requires precise finite element elastic calculations. We leave the discussion for future investigations. Moreover, it is important to notice that the linear broadening is not consistent with any leading parabolic behavior [8,13] on large scales where broadening increasing as a square root of depth would be expected. Therefore, our results clearly contradict the claim of generic parabolic behavior extracted from recent experiments [24] done on a quite specific granular assembly and obtained at a very small scale.

In conclusion, we present the first experimental determination of the horizontal stress distribution in response to a localized stress solicitation (i.e., the mechanical response function or Green's function) in a granular piling. The stress solicitations are made along the vertical axis and the spatial distribution of pressures is measured at the bottom of the pile. The different pilings tested were very disordered in terms of size, polydispersity, and friction between the grains. The piling procedure we use avoids, as much as possible, preparation memory effects. In such a case, the Green's function is consistent with predictions of elasticity since it does not exhibit the two side bumps as the hyperbolic modeling would. We also find a linear dependence of the half-height enlargement width depth. This last result rules out parabolic modeling of disordered granular assemblies (on a large scale, at least). Nevertheless, we do not find so far a complete quantitative agreement with the exact result of isotropic elasticity in an infinite half-space. An open question is still to understand how and whether this difference could be captured when considering explicitly boundary conditions imposed by the bottom plate and/or the explicit consideration of elastic anisotropy. This issue is left to a future report. In fact, recent experiments on bidimensional assemblies have shown that a change of shape of the response function can be related to an increase of packing disorder [25]. Interestingly this effect can be reproduced with hyperbolic models when strong disorder is included (see recent work by Bouchaud *et al.* [26]). Consequently, the goal of further experiments is to study in detail the influence of disorder and texture on the mechanical response function.

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