First Experimental Test of a Trace Formula for Billiard Systems Showing Mixed Dynamics

C. Dembowski,[1] H.-D. Gräf,[1] A. Heine,[1] T. Hesse,[2] H. Rehfeld,[1] and A. Richter[1]
[1]*Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*
[2]*Abteilung Theoretische Physik, Universität Ulm, D-89069 Ulm, Germany*
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In general, trace formulas relate the density of states for a given quantum mechanical system to the properties of the periodic orbits of its classical counterpart. Here we report for the first time on a semiclassical description of microwave spectra taken from superconducting billiards of the Limaçon family showing mixed dynamics in terms of a generalized trace formula derived by Ullmo *et al.* [Phys. Rev. E **54**, 136 (1996)]. This expression not only describes mixed-typed behavior but also the limiting cases of fully regular and fully chaotic systems and thus presents a continuous interpolation between the Berry-Tabor and Gutzwiller formulas.

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The semiclassical relationship between the density of states of a chaotic quantum system and the properties of the periodic orbits (POs) of the corresponding classical system has been known for nearly 30 years [1,2]. The so-called Gutzwiller trace formula expresses the density of states by a weighted sum over all individual classical POs. Integrable, i.e., regular, systems can be described by Einstein-Brillouin-Keller quantization [3]. A trace formula for such systems was first derived by Gutzwiller [4] and later in a different way by Berry and Tabor [5]. Besides these two limiting cases of chaotic and regular dynamics, systems with intermediate, mixed behavior have attracted more and more attention in recent years.

A very popular class of systems for these studies is Euclidean billiards, which are classically defined by the free motion of a particle inside a domain with elastic reflections at the boundary. The corresponding quantum billiard is described by the stationary Schrödinger equation with Dirichlet boundary conditions [6]. Two-dimensional billiards can be experimentally studied by using flat microwave resonators [7–9]. The aim of this Letter is to present for the first time an experimental test of a recently proposed trace formula by Ullmo *et al.* [10] for systems with mixed dynamics employing a set of such flat microwave billiards of the so-called Limaçon family [11].

This Letter is organized as follows: First, we recall the salient features of the trace formula for systems showing mixed behavior. Second, we follow this with a short description of the experimental techniques using superconducting microwave billiards which yield spectra of high resolution. Third, we present the application of the trace formula to the experimental data.

For the following brief discussion of the trace formula for mixed systems we restrict ourselves to systems with 2 degrees of freedom. We start with an integrable system, where the contribution of POs with topology $M = (M_1, M_2)$ specifying the individual winding number of the PO on the tori to the density of states is given by the Berry-Tabor formula [5]

$$\rho_M^{\mathrm{BT}}(S) = \frac{T}{\pi \hbar^{3/2} M_2^{3/2} |g_E''|^{1/2}} \cos\left(\frac{S}{\hbar} - \frac{\eta\pi}{2} - \frac{\pi}{4}\right), \tag{1}$$

with T being the period of the PO, g_E'' is the curvature of the line of constant energy $H(I_1, I_2) = E$, S is the action of the PO, and η is its Maslov index.

By moving from the regular to the near-integrable case, Ozorio de Almeida [12] added a small perturbation $\epsilon\mathcal{H}$ to an integrable system, described by the Hamiltonian H_0: $H(I,\phi) = H_0(I) + \epsilon\mathcal{H}(I,\phi)$, with (I,ϕ) denoting the action-angle variables. This small perturbation changes the density of states ρ_M^{BT} of the regular system in such a way that a first order correction to the action has to be added. This means that the resonant tori, on which the POs of the regular system exist, are being destroyed, and only two POs per torus will survive: one stable (s) and one unstable (u) PO (according to the Poincaré-Birkhoff theorem). Thus for near-integrable systems, Ozorio de Almeida found a modified Berry-Tabor expression for the density of states

$$\rho_M^{O}(S) = \rho_M^{\mathrm{BT}} J_0(\Delta S/\hbar), \tag{2}$$

where ΔS is the difference of the action of the stable and unstable PO.

In a typical case the unperturbed Hamiltonian H_0 and the perturbation $\epsilon\mathcal{H}$ in action-angle variables are not known. Thus, a generalization of formulas (1) and (2) and also a method to evaluate the parameters entering these formulas are needed.

Ullmo *et al.* [10] started their evaluation of a trace formula for mixed systems also with the Berry-Tabor expression. However, they did not use the propagator formalism of [5], but instead the energy dependent Green's function and also the result of Ozorio de Almeida [12]. However, they went one step farther. Instead of truncating the Fourier expansion of the corrected actions, which results in the damping Bessel term in Eq. (2), they mapped the problem onto the pendulum. They introduced an action which is a composition of the mean action $\overline{S}$

and the difference action ΔS of the two POs (stable and unstable), $\Delta S = (S_u - S_s)/2$ and $\overline{S} = (S_u + S_s)/2$. The actions S_u and S_s and also the monodromy matrices $\mathbf{M}_u$ and $\mathbf{M}_s$ of the two POs can be easily computed. Entering these relations into the integral which describes the dephasing of the PO contribution of the family M under a perturbation (see [10]), one is able to modify the expression of the density of states for the integrable case [Eq. (1)]. This yields the following expression for the density of states for each pair, the stable and the unstable PO:

$$\rho_M^U(S) = \frac{1}{\pi|\hbar^3 M_2^3 g_E''|^{1/2}} \mathrm{Re}\left\{\exp\left(\frac{i\overline{S}}{\hbar} - \frac{i\eta\pi}{2} - \frac{i\pi}{4}\right)\left[[\overline{T}[J_0(s) - i\tilde{a}J_1(s)] + i\Delta T\left[J_1(s) + \frac{i\tilde{a}}{2}[J_0(s) - J_2(s)]\right]\right]\right\}, \tag{3}$$

with $s = \Delta S/\hbar$ being the normalized correction to the action, $\overline{T}$ is the averaged period (half of the sum of two periods), and ΔT is their difference. The quantities $J_0(s)$, $J_1(s)$, and $J_2(s)$ are the standard Bessel functions. The value $\tilde{a}$ is the ratio of the determinants of the monodromy matrices of the stable and the unstable PO. For $\tilde{a} \to 0$, one obtains the result of Ozorio de Almeida [Eq. (2)]. The Maslov index is denoted by η, and for the evaluation of g_E'' see, e.g., [13].

To apply Eq. (3) to the billiard problem below, small replacements are necessary: The action is given by $S = \hbar kl$, with k being the wave number and l the length of the PO. The period of the PO can be expressed by its length and the term $M_2^3 g_E''$ can be evaluated by using expressions given in [10].

From Eq. (3) the two limiting cases, the Berry-Tabor result for integrable systems and the Gutzwiller result for chaotic systems, are easily reproduced. One obtains the first one for $\Delta S \to 0$ [Eq. (1)], while the other results from the asymptotic expression for the Bessel functions. Gutzwiller's trace formula reads as follows:

$$\rho^G(S) = \frac{1}{\pi\hbar}\sum_{\mathrm{PO}} \frac{T}{|\mathrm{Det}(\mathbf{M} - 1)|^{1/2}} \cos\left(\frac{S}{\hbar} - \eta\frac{\pi}{2}\right), \tag{4}$$

where $\mathbf{M}$ is the monodromy matrix and η is the Maslov index of the PO (see [1,2]). Ullmo *et al.* tested their trace formula numerically by applying it to a quartic oscillator for which they have calculated the first 12 000 eigenvalues. They found good agreement between the simulated quantum spectrum and its reconstruction with Eq. (3).

For a precise test of the trace formula for mixed systems [Eq. (3)] an accurate measurement of the resonances of the investigated microwave billiards is necessary. Therefore we studied experimentally a one-parameter family of superconducting two-dimensional microwave resonators. In Fig. 1 the shapes of the measured billiards are shown. They all belong to the family of Limaçon billiards, which have been numerically studied in [11]. Their boundary is defined as the quadratic conformal mapping of the unit disk onto the complex w plane: $w = z + \lambda z^2$, where $\lambda \in [0, 1/2]$ controls the chaoticity of the system. These billiards are also called Pascalian snails, their shape was already mentioned by the famous German painter, A. Dürer, in 1525 [14].

We have investigated in detail four billiards of different chaoticity with parameters $\lambda = 0$, $\lambda = 0.125$, $\lambda = 0.15$, and $\lambda = 0.3$. All billiards, except the first one, are desymmetrized. For $\lambda = 0$ we have a circle, which is known to be integrable, i.e., regular. Investigations of the classical Poincaré surface of section of the other configurations have shown, that the fraction of the chaotic phase space is 55% ($\lambda = 0.125$), 66% ($\lambda = 0.15$), and nearly 100% ($\lambda = 0.3$). This is supported by the statistical analysis (nearest neighbor spacing distribution and Dyson-Mehta statistics) of the microwave spectra [9].

The measurements were carried out in a liquid helium bath cryostat at $T = 4.2$ K [9]. The billiards were excited with frequencies up to 20 GHz, and a total number of more than 1000 resonances for each billiard (about 660 resonances for the circular billiard) were observed. The high quality factor of 10^5–10^7 together with the very good signal-to-noise ratio of up to 70 dB of the superconducting niobium cavities made it easy to separate the resonances from each other, especially in the case of the circular billiard with nearly twofold degenerate states. In Fig. 2 a part of the measured spectrum for the $\lambda = 0.15$ billiard is shown. The obtained eigenvalue sequences $\{k_1, k_2, \ldots, k_n\}$ form the basis for the following test of the trace formula for mixed systems.

The effect of the classical POs in the quantum mechanical system is seen in the Fourier transform of the fluctuating part of the level density

$$\tilde{\rho}^{\mathrm{fluc}}(l) = \int_{k_{\min}}^{k_{\max}} dk\, e^{ikl}[\rho(k) - \rho^{\mathrm{Weyl}}(k)], \tag{5}$$

where $\rho(k)$ is the measured level density of the system, $\rho^{\mathrm{Weyl}}(k)$ is its smooth part, and $[k_{\min}, k_{\max}]$ is the wave number interval in which the data were taken [15] (see Fig. 3). Note, that Eqs. (1), (3), and (4) already describe the fluctuating part of the level density.

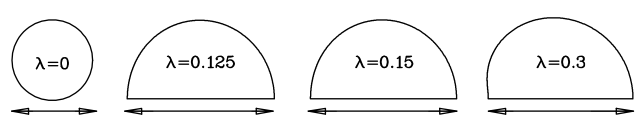

FIG. 1. Shapes of the investigated billiards of the Limaçon family. All billiards are desymmetrized, except the first one.

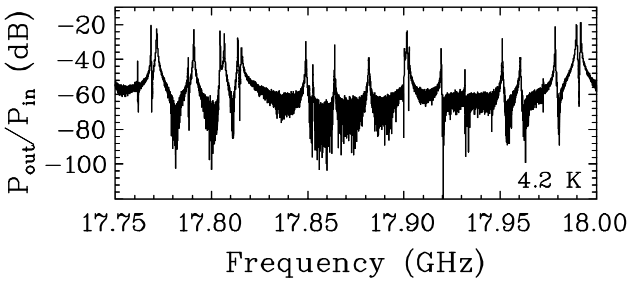

FIG. 2. Part of a transmission spectra of the superconducting microwave billiard with $\lambda = 0.15$ at $T = 4.2$ K.

By applying the trace formula Eq. (3) to the investigated systems, we restrict ourselves to the first POs up to a length of 1.4 m (see Fig. 3). Around this length (depending on each individual system) the family of the so-called *whispering gallery orbits* occurs. These orbits are characterized by twice the length of the outer circumference around which they are creeping. Thus the number of reflections go to infinity, leading to a broad peak in the Fourier spectrum caused by a large number of different POs, with more or less the same length, which interfere with each other. This makes a correct reconstruction of the measured spectrum with the help of Eq. (3), at this length very difficult.

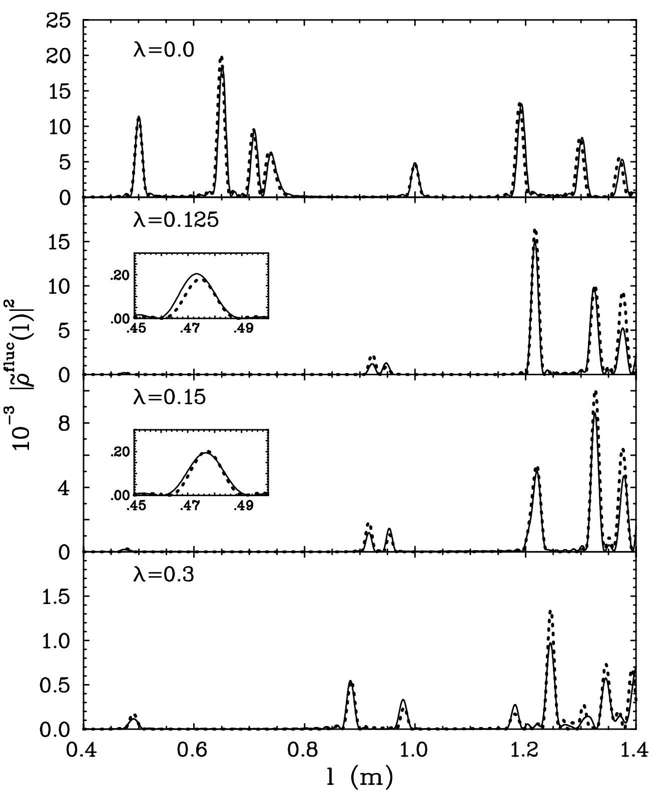

FIG. 3. Comparison between the measurement (solid lines) and the reconstruction (dashed lines) of length spectra with the help of the trace formula given by Eqs. (1), (3), and (4). The insets for the $\lambda = 0.125$ and $\lambda = 0.15$ billiards show a magnification of the first periodic orbit at $l \approx 0.47$ m.

The reconstruction of the spectrum of the circular billiard was done with the help of the Berry-Tabor formalism, one limiting case of Eq. (3), using a symbolic code which easily determines all POs [16]. For the three other billiards we calculated the properties of each PO (length, number of reflections, curvature of the boundary at the reflection point, and Maslov index) numerically. The so-found characteristic values for each PO form the basis for the reconstruction of the experimental spectra on the theoretical side.

A comparison between the experimental Fourier spectrum and its numerical reconstruction for the four investigated systems is presented in Fig. 3. For the circle the reconstruction is in very good agreement with the measurement. The reconstruction for the two billiards belonging to the mixed regime ($\lambda = 0.125$ and $\lambda = 0.15$ billiard) is also very satisfactory for the shortest POs, whereas for the following POs with lengths $l \geq 1.3$ m small deviations become visible. These deviations do not occur in the positions of the POs but in the height of the reconstructed peaks. The same situation is found for the chaotic $\lambda = 0.3$ billiard, where the predictions from Gutzwiller's trace formula were compared to the data.

In Fig. 4 the real and imaginary parts of the experimental and theoretical Fourier transformed fluctuating parts of the level density for the $\lambda = 0.125$ billiard around the periodic orbit pair at 1.21 m/1.22 m are compared. The theoretical reconstruction was calculated by using Eq. (3). The figure clearly displays that, besides the theoretical positions, the phases of the POs are also in good agreement with the experimental data.

The small deviations we found for the billiard systems with mixed dynamics, i.e., $\lambda = 0.125$ and 0.15, cannot be explained by the fact that we used *real* systems compared to the *ideal* systems we used for the reconstruction in Eq. (3). Numerical simulations [17] for those billiards have shown that our measured eigenfrequency

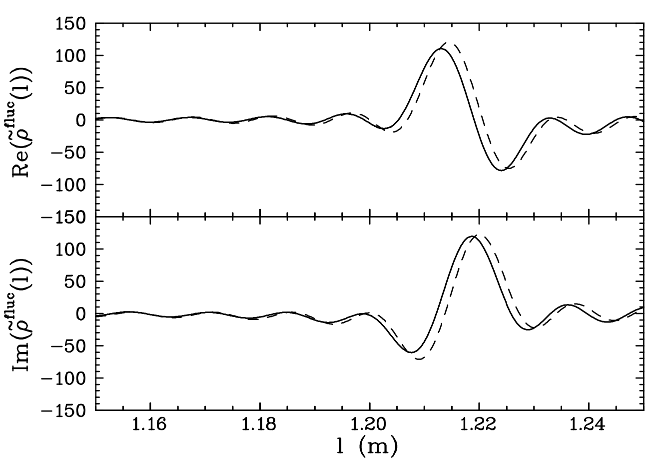

FIG. 4. Real and imaginary parts of the length spectrum of the $\lambda = 0.125$ billiard. The measured spectrum is drawn as solid lines and the reconstruction as dashed lines. The peaks belong to the periodic orbit pair at 1.21 m/1.22 m.

spectra are in very good agreement with the simulated eigenvalues. The most likely explanation for the deviations between theory and experiment is that Eq. (3) has been derived for the case of small perturbations of a regular system, i.e., for the near-integrable case, while the two investigated billiards already constitute highly mixed systems. Furthermore with only about 1000 measured eigenvalues we are probably still away from the true semiclassical regime. However, the results for the mixed systems obtained with the trace formula of Ullmo *et al.* is much more satisfying than using the Gutzwiller trace formula straightforwardly without taking the Poincaré-Birkhoff theorem into account. Equation (4) predicts the positions of the peaks in the Fourier spectrum correctly but fails badly on their amplitudes which are a measure of the stability of the POs. Finally, the small deviations found for the chaotic case ($\lambda = 0.3$) are due to the fact that for this special billiard we can see the differences between a *real* and an *ideal* system. In the manufacturing process the shaping of the boundary, in particular the cusp at the lower left corner of the billiard, is mechanically a real challenge. Note that especially the properties of the boundary, e.g., its curvature, determine the amplitude of the peak in the length spectrum. Comparing numerically simulated data [17] for the $\lambda = 0.3$ billiard with the Gutzwiller reconstruction [Eq. (4)] shows better agreement, so that we can conclude that the observed deviations for the chaotic billiard (lower part of Fig. 3) are indeed due to slight mechanical imperfections.

In summary, we have to our knowledge performed for the first time a reconstruction of an experimental length spectrum of billiards with mixed phase space dynamics. For this we used a recently derived trace formula which interpolates between the regular Berry-Tabor and the chaotic Gutzwiller cases. With the help of this formula [Eq. (3)], we were able to describe our measured data satisfactorily.

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