New Value of m_μ/m_e from Muonium Hyperfine Splitting

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The complete contribution to the muonium hyperfine splitting of relative order $\alpha^3(m_e/m_\mu)\ln\alpha$ is calculated. The result is much smaller than suggested by a previous estimate and leads to a 2σ upward shift of the most precise value for the muon-electron mass ratio. Analogous contributions are calculated for the positronium hyperfine splitting, where a discrepancy with experiment remains.

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Precise measurement of the ground-state muonium (μ^+e^-) hyperfine splitting (HFS) [1], together with the corresponding theoretical analysis, provides a stringent test of bound state theory in quantum electrodynamics (QED), and allows a precise determination of the fundamental physical constants m_μ/m_e and α. The theoretical prediction can be expressed as a series expansion in small parameters $\alpha \approx 1/137$ and $m_e/m_\mu \approx 1/207$; terms involving logarithms, $\ln\alpha^{-1} \approx \ln(m_\mu/m_e) \approx 5$, also appear. At leading order in α, the splitting is given by the Fermi energy ($Z = 1$ for muonium)

$$E_F = h\Delta\nu_F = \frac{16}{3}(hc)R_\infty Z^4\alpha^2\frac{m_e}{m_\mu}\left[1+\frac{m_e}{m_\mu}\right]^{-3}. \quad (1)$$

The complete splitting is the sum of terms [2,3]

$$\Delta\nu(\mathrm{Mu})_{\mathrm{th}} = \Delta\nu_D + \Delta\nu_{\mathrm{rad}} + \Delta\nu_{\mathrm{rec}} + \Delta\nu_{\mathrm{r\text{-}r}} + \Delta\nu_{\mathrm{weak}} + \Delta\nu_{\mathrm{had}}\,. \quad (2)$$

Here D stands for Dirac, or relativistic corrections, while the other terms are from radiative, recoil, radiative-recoil, weak, and hadronic contributions. Currently, theory is limited by uncalculated or imprecisely known terms in $\Delta\nu_{\mathrm{rec}}$ and $\Delta\nu_{\mathrm{r\text{-}r}}$ of order $E_F\alpha^3(m_e/m_\mu)$, some of which are enhanced by logarithmic factors; see Table I. This paper presents a calculation of terms of order $E_F\alpha^3(m_e/m_\mu)\ln\alpha$, with the following results:

$$\delta(\Delta\nu_{\mathrm{rec}}) = E_F\frac{(Z\alpha)^3}{\pi}\frac{m_e}{m_\mu}\ln(Z\alpha)^{-1} \times\left(\frac{101}{9} - 20\ln2\right), \quad (3)$$

$$\delta(\Delta\nu_{\mathrm{r\text{-}r}}) = E_F\frac{\alpha(Z\alpha)^2}{\pi}\frac{m_e}{m_\mu}\ln(Z\alpha)^{-1} \times\left(-\frac{431}{90} + \frac{32}{3}\ln2 + Z^2\right). \quad (4)$$

Numerically, these contributions give $-0.034 + 0.047 = 0.013$ kHz, much smaller than the $-0.263(60)$ kHz suggested by a previous incomplete calculation [4,5]. A discussion of the error due to still uncalculated terms is given at the end of the paper.

Including the complete results of Eqs. (3) and (4) does not significantly alter the good agreement between the experimental value [1] and the theoretical prediction [2] for the HFS in physical units, since the error in the latter is due mainly to the measured value of m_μ/m_e. Likewise, the HFS determination of α is not significantly changed [2]. However, the new results in Eqs. (3) and (4) represent a fractional shift of 6.2×10^{-8} in the HFS, and hence also in the HFS determination of the mass ratio m_μ/m_e. Also, the order $(Z\alpha)^2$ part of $\Delta\nu_{\mathrm{rec}}$ used in Ref. [2] contains only the leading m_r^2/m_em_μ piece; the remaining piece contributes an additional 0.065(6) kHz or 1.5×10^{-8} [6]. The mass ratio is then shifted from the value in Ref. [2] [Eq. (161)] to become

$$(m_\mu/m_e)[\Delta\nu(\mathrm{Mu})] = 206.768\,281\,7(33)\,(24)\,(16)\,[2.1\times10^{-8}], \quad (5)$$

with the errors arising from uncertainty in $\Delta\nu_{\mathrm{theory}}$ due to uncalculated terms, from $\Delta\nu_{\mathrm{expt}}$, and from the value of α, respectively. This represents a shift of 2.5σ (in terms of the previous relative error 3.1×10^{-8}) and a 30% reduction in error.

The positronium (e^+e^-) HFS has also been measured precisely, though at present its interest is for testing our knowledge of QED bound states, as opposed to determining fundamental constants. The two most precise values

TABLE I. Contributions of order $E_F\alpha^3(m_e/m_\mu)$ to the muonium HFS. The second column lists the contributions used in Ref. [2]; the third column gives new or modified values from the present paper. Asterisks denote partial results.

$\times E_F\frac{m_e}{m_\mu}$	Ref. [2] (kHz)	This paper (kHz)
$(Z\alpha)^3\ln^2(Z\alpha)$	−0.043	
$(Z\alpha)^3\ln(Z\alpha)\ln(m_\mu/m_e)$	−0.210	
$(Z\alpha)^3\ln(Z\alpha)$	−0.257(*)	−0.034
$(Z\alpha)^3\ln(m_\mu/m_e)$	···	−0.035 (*) [21]
$(Z\alpha)^3$	0.107(30)	
$\alpha(Z\alpha)^2\ln^2(Z\alpha)$	0.344	
$\alpha(Z\alpha)^2\ln(Z\alpha)$	−0.008 (*)	0.034
$\alpha(Z\alpha)^2$	−0.107(30)	
$Z^2\alpha(Z\alpha)^2\ln(Z\alpha)$	···	0.013
$\alpha^2(Z\alpha)\ln^3(m_\mu/m_e)$	−0.055	
$\alpha^2(Z\alpha)\ln^2(m_\mu/m_e)$	0.010	
$\alpha^2(Z\alpha)\ln(m_\mu/m_e)$	0.009 (*)	
$\alpha^2(Z\alpha)$	···	

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are due to Mills and Bearman [$\Delta\nu(P)_{\rm expt1}$, Ref. [7]] and Ritter *et al.* [$\Delta\nu(P)_{\rm expt2}$, Ref. [8]],

$$\Delta\nu(\mathrm{Ps})_{\rm expt1} = 203\,387.5(1.6)\ \mathrm{MHz}\ [7.9 \times 10^{-6}], \qquad (6)$$

$$\Delta\nu(\mathrm{Ps})_{\rm expt2} = 203\,389.10(74)\ \mathrm{MHz}\ [3.6 \times 10^{-6}]. \qquad (7)$$

The theoretical expression is

$$\Delta\nu(\mathrm{Ps})_{\rm th} = m_e\alpha^4\left[C_0 + C_1\frac{\alpha}{\pi} + \alpha^2(C_{21}\ln\alpha^{-1} + C_{20}) + \frac{\alpha^3}{\pi}(C_{32}\ln^2\alpha^{-1} + C_{31}\ln\alpha^{-1} + C_{30}) + \mathcal{O}(\alpha^4)\right]. \quad (8)$$

Including the known terms through C_{20} [9] yields $\Delta\nu(\mathrm{Ps})_{\alpha^2} = 203\,392.93$ MHz. Coefficient $C_{32} = -7/8$ has been known for some time [10], and in this paper we calculate

$$C_{31} = 217/90 - 17\ln 2/3\,. \qquad (9)$$

C_{32} and C_{31} contribute -0.91 MHz and -0.32 MHz to $\Delta\nu(\mathrm{Ps})_{\rm theory}$, respectively, bringing the theoretical prediction to

$$\Delta\nu(\mathrm{Ps})_{\rm th} = 203\,391.70(20)\ \mathrm{MHz}\ [1.0 \times 10^{-6}]. \qquad (10)$$

The uncertainty of 0.20 MHz corresponds to a coefficient $C_{30} \approx 4$. The discrepancy with experiment is significant: 2.5σ and 3.5σ for Eqs. (6) and (7), respectively. As with the orthopositronium lifetime [11,12], a true disagreement between experiment and the predictions of QED would have important consequences.

The calculation is done in the framework of an effective quantum mechanical Hamiltonian theory [12], taking inputs from relativistic QED field theory and from (nonrelativistic) NRQED field theory [13]. The results to be derived for muonium can be translated directly to positronium by taking $m_\mu \to m_e$, and including the additional contributions from virtual e^+e^- annihilation.

The Hamiltonian can be decomposed into the sum

$$H = H_0 + V_4 + V_5 + V_6 + V_7\,, \qquad (11)$$

where H_0 is the unperturbed Hamiltonian for the Coulomb problem with reduced mass $m_r = m_e m_\mu/(m_e + m_\mu)$,

$$H_0 = \frac{p^2}{2m_r} - \frac{(Z\alpha)}{r}\,. \qquad (12)$$

Potentials V_4, V_5, V_6, and V_7 give contributions to the energy of order $m\alpha^4$, $m\alpha^5$, etc. Since non-HFS operators will affect the HFS only in second- or higher-order perturbation theory, it follows that only the HFS parts of potentials V_6 and V_7 are necessary. Furthermore, any potential not contributing to S states (in first- or second-order perturbation theory) may be neglected. We will write the potentials in terms of a list of standard operators ($q \equiv l - k$):

$$\begin{aligned}
\langle l|\mathcal{O}_1|k\rangle &= \frac{1}{m_r^2}\,,\\
\mathcal{O}_2 &= \frac{1}{\pi(Z\alpha)m_r^2}\,p^i\left(\frac{p^2}{2m_r} - \frac{Z\alpha}{r} - E\right) \times \ln\frac{m_r/2}{\frac{p^2}{2m_r} - \frac{Z\alpha}{r} - E}\,p^i\,,\\
\langle l|\mathcal{O}_3|k\rangle &= \frac{1}{m_r^2}\ln\frac{|q|}{m_r}\,, \quad \langle l|\mathcal{O}_4|k\rangle = \frac{1}{m_r^2}\frac{|l\times k|^2}{q^2}\,,\\
\langle l|\mathcal{O}_5|k\rangle &= \pi(Z\alpha)\frac{|q|}{m_r}\,, \quad \langle l|\mathcal{O}_6|k\rangle = \frac{q^2}{m_r^2}\ln\frac{|q|}{m_r}\,,\\
\mathcal{O}_7 &= \frac{1}{\pi(Z\alpha)}\frac{p^4}{m_r^3}\,, \quad \langle l|\mathcal{O}_8|k\rangle = \frac{1}{m_r^2}\frac{|l\times k|^2}{q^4}\,,\\
\langle l|\mathcal{O}_9|k\rangle &= \frac{1}{m_r^2}\left(\sigma_e\cdot\sigma_\mu - \frac{3q\cdot\sigma_e q\cdot\sigma_\mu}{q^2}\right). \qquad (13)
\end{aligned}$$

Potential V_4 is derived from tree-level NRQED diagrams with Fermi, Darwin, kinetic, and dipole vertices [14], and contains the leading relativistic corrections,

$$\frac{V_4}{4\pi(Z\alpha)} = \left\{\frac{c_F^e c_F^\mu}{6}\frac{m_r^2}{m_e m_\mu}\left[\sigma_e\cdot\sigma_\mu\mathcal{O}_1 + \frac{1}{2}\mathcal{O}_9\right] + \frac{1}{8}\left(c_D^e\frac{m_r^2}{m_e^2} + c_D^\mu\frac{m_r^2}{m_\mu^2}\right)\mathcal{O}_1 - \frac{1}{32}\left(\frac{m_r^3}{m_e^3} + \frac{m_r^3}{m_\mu^3}\right)\mathcal{O}_7 - \frac{m_r^2}{m_e m_\mu}\mathcal{O}_8\right\}. \qquad (14)$$

Renormalization constants $c_F \approx c_D = 1 + \mathcal{O}(\alpha)$ are tabulated below. V_5 gives the leading radiative corrections,

$$\begin{aligned}
\frac{V_5}{(Z\alpha)} = {} & \frac{2\alpha}{3}\left(\frac{m_r^2}{m_e^2} + 2Z\frac{m_r^2}{m_e m_\mu} + Z^2\frac{m_r^2}{m_\mu^2}\right)\mathcal{O}_2 + \frac{14(Z\alpha)}{3}\frac{m_r^2}{m_e m_\mu}\mathcal{O}_3\\
& + \left\{-\frac{4\alpha}{3}\left(\frac{m_r^2}{m_e^2}\ln\frac{m_r}{m_e} + Z^2\frac{m_r^2}{m_\mu^2}\ln\frac{m_r}{m_\mu}\right) - \frac{4\alpha}{15}\frac{m_r^2}{m_e^2} + (Z\alpha)\frac{m_r^2}{m_e m_\mu}\right.\\
& \left.\times\left[\frac{-2}{m_\mu^2 - m_e^2}\left(m_\mu^2\ln\frac{m_e}{m_r} - m_e^2\ln\frac{m_\mu}{m_r}\right) + \frac{20}{9} - \frac{2m_e m_\mu}{m_\mu^2 - m_e^2}\ln\frac{m_\mu}{m_e}\sigma_e\cdot\sigma_\mu\right]\right\}\mathcal{O}_1\,. \qquad (15)
\end{aligned}$$

The contribution from muon vacuum polarization is not relevant to our analysis, and has been excluded from V_5. For V_6, only HFS terms are necessary. These again are taken directly from NRQED diagrams:

$$\frac{V_6}{4\pi(Z\alpha)} = \frac{\sigma_e \cdot \sigma_\mu}{m_e m_\mu}\left\{\left[\frac{m_r^2}{m_e m_\mu}\left(\frac{c_S^e c_S^\mu}{48} + \frac{c_F^e c_F^\mu}{6} - \frac{c_F^e c_S^\mu + c_S^e c_F^\mu}{12}\right) - \frac{1}{24}\left(c_{p'p}^e c_F^\mu \frac{m_r^2}{m_e^2} + c_F^e c_{p'p}^\mu \frac{m_r^2}{m_\mu^2}\right)\right]\mathcal{O}_4 - \frac{1}{48}\left(c_S^e c_F^\mu \frac{m_r}{m_e} + c_F^e c_S^\mu \frac{m_r}{m_\mu}\right)\mathcal{O}_5\right\}. \tag{16}$$

Spin-orbit, retardation, time-derivative, $p'p$, and Seagull interactions have been included. Additional local operator terms, of the form $-\nabla^2\delta^3(r)$ and $\{p^2, \delta^3(r)\}$, are not shown explicitly; these analytic terms do not generate factors of $\ln\alpha$, and so are not relevant to the present analysis. The necessary renormalization constants have already been calculated [14,15],

$$c_F^e = 1 + a_e, \qquad c_D^e = 1 + \frac{8\alpha}{3\pi}\left(-\frac{3}{8} + \frac{5}{6}\right) + 2a_e,$$
$$c_S^e = 1 + 2a_e, \qquad c_{p'p}^e = a_e\,. \tag{17}$$

Here $a_e = \alpha/2\pi + \mathcal{O}(\alpha^2)$ is the electron anomalous magnetic moment. For c^μ, m_μ and $Z^2\alpha$ are substituted for m_e and α.

Potential V_7 has no noninstantaneous HFS contribution coming from photon momenta $q \approx m\alpha^2$, a consequence of the fact that spin-dependent $M1$ multipole transitions vanish in the absence of relativistic effects, and are therefore suppressed. The remaining instantaneous part of V_7, from momenta $q \approx m\alpha$, is fully determined by requiring that the Hamiltonian correctly reproduces the low-momentum expansion of the one-loop photon-exchange scattering amplitude. Introducing a photon mass λ, and ultraviolet cutoff Λ on photon momenta, the effective Hamiltonian (without V_7) gives [16]

$$\frac{2(Z\alpha)^2 q^2}{3m_e^2 m_\mu^2}\,\sigma_e \cdot \sigma_\mu\left(-\frac{2}{3}\ln\frac{\Lambda}{\lambda} + \cdots\right), \tag{18}$$

where again analytic terms are not shown. The corresponding QED amplitude is

$$\frac{2(Z\alpha)^2 q^2}{3m_e^2 m_\mu^2}\,\sigma_e \cdot \sigma_\mu\left(-\frac{2}{3}\ln\frac{\Lambda}{\lambda} + \frac{1}{4}\log\frac{q}{\Lambda} + \ldots\right). \tag{19}$$

This result has been checked both in QED Feynman gauge, and in NRQED Coulomb gauge. Requiring the effective theory to match QED implies that

$$V_7 = \frac{(Z\alpha)^2}{6}\,\frac{m_r^2}{m_e^2 m_\mu^2}\,\sigma_e \cdot \sigma_\mu \mathcal{O}_6\,. \tag{20}$$

Contributions to V_7 having a dependence on m_e, m_μ, α, and Z different from Eq. (20) are ruled out by noticing the following: (i) The nonrecoil contributions are already present in V_4, V_5, and V_6 (as we will soon verify), so that V_7 contains no nonrecoil piece; and (ii) masses can enter only as inverse powers $1/m_e$ and $1/m_\mu$, and, in particular, not as $1/(m_e + m_\mu)$. This latter result can be seen clearly using time ordered perturbation theory in NRQED: the NRQED vertices are all homogeneous in the masses. Furthermore, the energy denominators will all have the form $1/(|q| + p_1^2/m_e + p_2^2/m_\mu)$, with photon momentum q and particle momenta p_1, p_2. (Contributions which are not simply iterations of lower-order potentials must have at least one photon in each intermediate state.) Such an expression, for $q \approx p_1 \approx p_2 \approx m\alpha$, can be expanded in powers of $p_1^2/m_e|q|$, $p_2^2/m_\mu|q|$—again homogeneous in the masses. Using (i) and (ii), the only parameter dependence which is symmetric in m_e and m_μ is that of Eq. (20).

Having completed the specification of the Hamiltonian, Eqs. (11), (12), (14), (15), (16), and (20), we now use the usual expressions from Rayleigh-Schrödinger perturbation theory to solve for the energy shift:

$$\Delta E = \langle V_6 + V_7\rangle + 2\langle V_4 \tilde{G} V_5\rangle + \langle V_4\rangle\left\langle\frac{\partial V_5}{\partial E}\right\rangle, \tag{21}$$

where $\tilde{G}$ is the Coulomb Green's function with ground-state pole removed, and $\langle V\rangle$ is the expectation value of V in the ground state of the unperturbed H_0, Eq. (12). The logarithmic contributions of the necessary matrix elements are

$$\frac{\langle\mathcal{O}_i\rangle}{\langle\delta^3(r)\rangle} \to (Z\alpha)^2\ln(Z\alpha)^{-1}\begin{cases}2, & i = 4,\\ 8, & i = 5,\\ 12, & i = 6,\end{cases} \qquad \frac{\langle\mathcal{O}_9\tilde{G}\mathcal{O}_9\rangle}{\langle\delta^3(r)\rangle} \to \frac{10}{m_r^2}\,\frac{(Z\alpha)}{\pi}\ln(Z\alpha)^{-1},$$
$$\left(2\frac{\langle\mathcal{O}_i\tilde{G}\delta^3(r)\rangle}{\langle\delta^3(r)\rangle} + \left\langle\frac{\partial\mathcal{O}_i}{\partial E}\right\rangle\right) \to \frac{(Z\alpha)}{\pi}\ln(Z\alpha)^{-1} \times \begin{cases}-2, & i = 1,\\ -4\ln(Z\alpha)^{-1} + 6 - 8\ln 2, & i = 2,\\ \ln(Z\alpha)^{-1} + 1 - 2\ln 2, & i = 3,\\ -16, & i = 7,\\ -1, & i = 8,\end{cases} \tag{22}$$

where the arrows signify that only logarithmic corrections, and in the case of $\langle\mathcal{O}_9\tilde{G}\mathcal{O}_9\rangle$, only the HFS part, are shown. The pure recoil result for the HFS at order $E_F(Z\alpha)^3(m_e/m_\mu)$ contains the previously known $\ln^2(Z\alpha)$ and $\ln(Z\alpha)\ln(m_\mu/m_e)$ contributions [10,14,17]; the new $\ln(Z\alpha)$ term is shown in Eq. (3) [18]. For radiative corrections at order $E_F\alpha(Z\alpha)^2$, the nonrecoil $\ln^2(Z\alpha)$ and $\ln(Z\alpha)$ terms, and the recoil $(m_e/m_\mu)\ln^2(Z\alpha)$ term [10], agree with previous calculations. A part of the radiative-recoil single logarithm was included previously [4,5]; the complete contribution is given in Eq. (4). Numerical values are summarized in Table I.

For positronium, there are additional interactions due to virtual annihilation of the electron and positron. The hard annihilation process is described by local operators, which by themselves cannot generate nonanalytic factors. So, for $\ln\alpha$ contributions, only second-order perturbations involving V_4 and V_5 need be considered:

$$\delta V_4 = \frac{\pi\alpha}{2}\left(\frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4}\right)\mathcal{O}_1\,, \tag{23}$$

$$\delta V_5 = \alpha^2\left[\left(-\frac{22}{9}\right)\left(\frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4}\right) + (-1 + \ln 2)\left(\frac{1}{4} - \frac{\sigma_e \cdot \sigma_\mu}{4}\right)\right]\mathcal{O}_1\,. \tag{24}$$

δV_4 gives the leading contribution from one-photon annihilation. The first and second terms of δV_5 come from radiative corrections to δV_4, and from two-photon virtual annihilation, respectively. $\mathcal{O}(m\alpha^7 \ln\alpha)$ contributions from these annihilation operators are

$$\delta(\Delta\nu_{\rm ann}) = m_e \frac{\alpha^7}{\pi}\ln\alpha^{-1} \times \left(-\frac{3}{8}\ln\alpha^{-1} + \frac{2261}{1080} - 3\ln 2\right). \tag{25}$$

The nonannihilation contributions for positronium are obtained by taking the limit $m_\mu \to m_e$ in the muonium analysis (making no expansion in m_e/m_μ); the combined result is given in Eq. (9).

The previously most significant sources of error in the muonium HFS were $\Delta\nu_{\text{r-r}}$ (0.104 kHz) and $\Delta\nu_{\rm rec}$ (0.060 kHz) [2]; all other uncertainties are estimated below 0.010 kHz [19]. Since there are still uncalculated terms at $\mathcal{O}(E_F\alpha^2(Z\alpha)(m_e/m_\mu)\ln(m_\mu/m_e))$ [20] and $\mathcal{O}(E_F\alpha(Z\alpha)^2 m_e/m_\mu)$, we take the uncertainty in $\Delta\nu_{\text{r-r}}$ as 0.040 kHz. The uncertainty in $\Delta\nu_{\rm rec}$ should remain approximately the same, since it is dominated by the still uncalculated terms of order $\mathcal{O}(E_F(Z\alpha)^3(m_e/m_\mu)\ln(m_\mu/m_e))$ [21] and $\mathcal{O}(E_F(Z\alpha)^3(m_e/m_\mu))$. Thus we take 0.070 kHz as an estimate of the total remaining theoretical error.

In the final stages of the calculation, the author received word from Melnikov and Yelkhovsky that they have also performed the calculation of $\alpha^3\ln\alpha$ terms, in a dimensional regularization approach [22]. The agreement found in different formalisms in two independent calculations lends strong support to the correctness of the results.

This work was motivated in part by, and is an extension of, Ref. [12]. Many ideas used in the calculation originated with G. P. Lepage, who the author thanks for continued insights and encouragement. Thanks are also due to P. Labelle, and to K. Melnikov and A. Yelkhovsky for useful conversations. This work was supported by a grant from the National Science Foundation.

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