From Computation to Black Holes and Space-Time Foam

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We show that quantum mechanics and general relativity limit the speed ν of a *simple* computer (such as a black hole) and its memory space I to $I\nu^2 \lesssim t_P^{-2}$, where t_P is the Planck time. We also show that the lifetime of a *simple* clock and its precision are similarly limited. These bounds and the holographic bound originate from the same physics that governs the quantum fluctuations of space-time. We further show that these physical bounds are realized for black holes, yielding the correct Hawking black hole lifetime, and that space-time undergoes much larger quantum fluctuations than conventional wisdom claims—almost within range of detection with modern gravitational-wave interferometers.

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The past few decades have witnessed amazing growth in the ability and speed with which computers can process information. Quantum computation only adds to the prospect that this exponential growth in information processing power will continue. But it is natural to ask whether this growth can go on indefinitely or whether there are physical laws that impose limitations to it [1,2]. In this Letter we show that indeed the laws of quantum mechanics and gravitation put considerable bounds on computation. In particular, the number ν of operations per unit time and the number I of bits of information in the memory space of a simple computer (simple in the sense to be made precise below) are both limited by the input power such that their product is bounded by a universal constant given by $I\nu^2 \lesssim t_P^{-2}$, where $t_P = (\hbar G/c^5)^{1/2}$ is the Planck time formed by the speed of light c, the quantum scale $\hbar$, and the gravitational constant G. Along the way, we also show that the total running time T over which a simple clock can remain accurate and the smallest time interval t that the clock is capable of resolving are bounded by $T \lesssim t(t/t_P)^2$. Interestingly, these bounds are saturated for black holes. So black holes, in some sense, may be regarded as the ultimate simple computers and ultimate simple clocks (though it may be extremely difficult or even impossible to realize this technological feat). As a demonstration of the unity of physics, we show that the physics that sets the limits to computation is precisely the physics that governs the quantum fluctuations of space-time [3,4] which, as pointed out recently [5–7], can plausibly be detected with gravitational-wave interferometers such as LIGO/VIRGO and LISA through future refinements. Furthermore, the same physics underlies the holographic principle. Thus the physics behind simple clocks, simple computers, black holes, space-time foam, and the holographic principle is interrelated. It is this interrelationship that we emphasize in this Letter.

The ingredients we use to derive the physical limits to computation are the general principles of quantum mechanics and general relativity which should suffice for the physics in the low-energy regime of quantum gravity in which we are interested. [Thus, in what follows, it is understood that all the time intervals we are dealing with are much greater than the Planck time, and all the distances much larger than the Planck length (ct_P).] Following Wigner [8], one can use quantum mechanics to set fundamental limits on the mass m of any system that serves as a time-registering device. Briefly, the argument goes as follows: If the clock has a linear spread of δR, then its momentum uncertainty is $\hbar(\delta R)^{-1}$. After a time τ, its position spread grows to $\delta R(\tau) = \delta R + \hbar\tau m^{-1}(\delta R)^{-1}$ with the minimum at $\delta R = (\hbar\tau/m)^{1/2}$. At the end of the total running time T, the linear spread can grow to

$$\delta R \gtrsim \left(\frac{\hbar T}{m}\right)^{1/2}. \tag{1}$$

But for the clock to give time to within accuracy t, it must have a small enough spread in position, so small that the time at which a light quantum strikes it (in order to read the time) can be determined within the required accuracy t; thus $\delta R \lesssim ct$. In other words, we require the wave packet of the center of mass of the clock be confined, throughout the running time T, to a region of the size ct. It follows that, for a given T and t, the lower bound on m reads

$$m \gtrsim \frac{\hbar}{c^2 t}\left(\frac{T}{t}\right). \tag{2}$$

This limit is more restrictive than that given by Heisenberg's energy-time uncertainty relation because it requires repeated measurement of time not to introduce significant inaccuracies over the total running time T. This is one of two reasons [the other one being Eq. (3) below] that the physical limits to computation we derive below are *more* restrictive than what one may expect [1].

Next, as shown by the present author and van Dam [3], one can supplement the quantum mechanical relation Eq. (2) with a fundamental limit from general relativity. In essence one finds that the minimum time interval that a clock can be used to measure is the light travel time across its Schwarzschild radius. The argument is quite

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simple. Let the clock be a simple light-clock consisting of two parallel mirrors (each of mass $m/2$) between which bounces a beam of light. On the one hand, for the clock to be able to resolve a time interval as small as t, the mirrors must be separated by a distance d with $d/c \lesssim t$. On the other hand, d is necessarily larger than the Schwarzschild radius Gm/c^2 of the mirrors so that the time registered by the clock can be read off at all. From these two requirements, it follows that the upper bound on m is given by [3]

$$t \gtrsim \frac{Gm}{c^3}\,. \tag{3}$$

As clocks, black holes saturate this bound (more on this later).

One can now use Eq. (2) to obtain a bound on the speed of computation ν of any information processor [9]. The mean input power given by $P = mc^2/T$ and the fastest possible processing frequency given by $\nu = t^{-1}$ are bounded [via Eq. (2)] as

$$\nu^2 = \frac{1}{t^2} \lesssim \frac{mc^2}{\hbar T} = \frac{P}{\hbar}\,. \tag{4}$$

Thus power limits speed of computation.

Next, by substituting Eq. (2) into Eq. (3) we can relate T to t as

$$T \lesssim t\left(\frac{t}{t_P}\right)^2. \tag{5}$$

Thus the better precision a clock attains, i.e., the smaller t is, the shorter it can keep accurate time, i.e., the smaller T is. With the Planck time being only about 10^{-43} sec, this bound on T is of no practical consequence. For example, a femtosecond (10^{-15} sec) precision yields the bound $T \lesssim 10^{34}$ years.

Now it is time for us to make precise what we mean by the qualification "simple" characterizing the simple clock and the simple computer. Alert readers may have already questioned the validity of Eq. (3), and accordingly also of the above T-t relation [Eq. (5)]. For example, consider a large clock consisting of N identical small clocks to keep time one after another. For large enough N, the T-t relation and Eq. (3) are violated for the large clock. But note that this argument is not valid if we consider only those clocks for which no such separation of components is involved. They are what we call *simple* clocks. The same qualification will be understood to apply to *simple* computers. The origin of this qualification can be traced to the bound given by Eq. (3). Why should we be interested in simple clocks and simple computers? For the simple reason that nature makes use of them. (This point is made clear below when we derive the holographic principle and when we discuss the case of black holes.)

Let us use the T-t relation in Eq. (5) to put a limit on the memory space of a computer. The point is that T/t, the maximum number of steps of information processing, is, aside from factors like ln2, the amount of information I that can be registered by the computer. With the aid of Eq. (4), the T-t relation yields

$$I \sim \frac{T}{t} \lesssim \frac{1}{(\nu t_P)^2} \sim \frac{\hbar}{P t_P^2}\,. \tag{6}$$

While it is not too surprising that the input power P limits the speed of computation ν [as given by Eq. (4)], it is less expected that power also limits memory space of a computer in the way given by Eq. (6). We note that Eq. (2) and Eq. (3) can also be used to give $I\nu \lesssim mc^2/\hbar$ and $\nu \lesssim \hbar/(t_P^2 mc^2)$, respectively.

More interestingly, Eq. (6) shows that the product of I and ν^2 is bounded by a universal constant

$$I\nu^2 \lesssim \frac{1}{t_P^2} \equiv \frac{c^5}{\hbar G} \sim 10^{86}\ \mathrm{sec}^{-2}, \tag{7}$$

independent of the mass, size, and details of the simple computer. For the numerical value in Eq. (7), we have used the speed of light in vacuum for c in t_P. This expression (valid for simple computers) links together our concepts of information, gravity, and quantum uncertainty. We see below that nature seems to respect this bound which, in particular, is realized for black holes. The restriction to simple computers is the price we have to pay for the universality of this bound. For comparison, current laptops perform about 10^{10} operations per second on 10^{10} bits, yielding $I\nu^2 \sim 10^{30}$ sec^{-2}.

As intriguing as the physical limits to computation are, it is perhaps even more amazing that the physics behind them is also what governs the quantum fluctuations of space-time. To see this, let us consider measuring the distance $R \gg ct_P$ between two points. We can put a clock at one of the points and a mirror at the other point. By sending a light signal from the clock to the mirror in a timing experiment we can determine the distance. But the quantum uncertainty in the positions of the clock and the mirror introduces an inaccuracy δR in the distance measurement. The same argument used above to derive the T-t relation now yields a similar bound for δR:

$$\delta R\left(\frac{\delta R}{ct_P}\right)^2 \gtrsim R\,, \tag{8}$$

in a distance measurement [3,10]. In a time measurement, an analogous bound is given by Eq. (5) with T playing the role of the measured time and t the uncertainty [3]. This limitation to space-time measurements can be interpreted as resulting from quantum fluctuations of space-time itself. In other words, at short distance scales, space-time is foamy. Thus the same physics underlies both the foaminess of space-time and the limits to computation and clock precision. Not surprisingly, these bounds have the same form. It is remarkable that modern gravitational-wave interferometers, through future refinements, may reach displacement noise levels low enough to test this space-time foam model [3,4], because the intrinsic foaminess of space-time provides another source of noise in the interferometers that can be highly constrained experimentally [5,7]. According to one estimate [5,7], if Eq. (8) is correct, the "advanced phase" of LIGO is expected to achieve a noise level low enough to probe t_P down to 10^{-41} sec, only about 2 orders of magnitude from what we expect it ($t_P \sim 10^{-43}$ sec) to be.

Furthermore, the same physics is behind the holographic principle, which states that the number of degrees of freedom of a region of space is bounded (not by the volume but) by the area of the region in Planck units [11]. To see this, consider a region of space with linear dimension R. Conventional wisdom claims that the region can be partitioned into cubes as small as $(ct_P)^3$. It follows that the number of degrees of freedom of the region is bounded by $(R/ct_P)^3$, i.e., the volume of the region in Planck units. But according to Eq. (8), the smallest cubes into which we can partition the region cannot have a linear dimension smaller than $(Rc^2t_P^2)^{1/3}$. Therefore, the number of degrees of freedom of the region is bounded by $[R/(Rc^2t_P^2)^{1/3}]^3$, i.e., the area of the region in Planck units, as stipulated by the holographic principle [7]. [A judicious application of Eq. (6) can also yield this result.] Thus the holographic principle has its origin in the quantum fluctuations of space-time. Turning the argument around, we believe the holographic principle alone suggests that the quantum fluctuations of space-time are as given by Eq. (8) [$\delta R \gtrsim (Rc^2t_P^2)^{1/3}$] and hence are much larger than what conventional wisdom [12] leads us to believe ($\delta R \gtrsim ct_P$) [13].

Let us now ask for what kind of physical systems are the physical limits listed above saturated (order of magnitudewise). There is at least one (and very likely only one) such system: the system of black holes. Since black holes have an entropy given by the event horizon area in Planck units [14], the holographic bound is obviously realized. [Alternatively, we can show this by using Eqs. (9) and (10) below.] Next, consider a black hole as a clock; then it is reasonable to expect that the maximum running time of this gravitational clock is given by the Hawking black hole lifetime

$$T \sim T_H \equiv \frac{G^2m^3}{\hbar c^4} \tag{9}$$

and that the minimum interval that the black hole can be used to measure is given by the light travel time across the black hole's horizon

$$t \sim \frac{Gm}{c^3}. \tag{10}$$

It is interesting that both Eqs. (9) and (10) can actually be derived by appealing to Wigner's two inequalities Eqs. (1) and (2) and using the Schwarzschild radius of the black hole as the minimum clock size [9]. Better yet, if we use Eqs. (2) and (3), we can immediately find that the bound on T is given by the Hawking black hole lifetime. Thus, if we had not known of black hole evaporation, this remarkable result would have implied that there is a maximum lifetime for a black hole. Now note that according to Eq. (10), the limit on t as shown in Eq. (3) is saturated for a black hole. Furthermore, using Eqs. (9) and (10) one can easily show that the bound given by Eq. (2) is saturated. It then follows that all the subsequent bounds [from Eqs. (4)–(7)] are saturated for black holes. As a check, we can combine Eqs. (9) and (10) to yield $T/t^3 \sim t_P^{-2}$, which saturates the T-t bound given in Eq. (5). On the other hand, when a black hole is considered as an information processor with power $P = mc^2/T_H \sim \hbar c^6/G^2m^2$, we can use Eq. (10) to obtain $\nu^2 \sim P/\hbar$, which realizes the bound given by Eq. (4). Equations (9) and (10) can also be used to yield $I \sim T/t \sim \hbar/Pt_P^2$, which saturates the bound given by Eq. (6). Finally, with both ν- and I-bounds saturated, the universal bound on computation given by Eq. (7) is also saturated for black holes. All these results reinforce the conceptual importance of black holes as the simplest and most fundamental [15] constructs of space-time, which set the universal limits to computation, clock precision, and numbers of degrees of freedom. These properties of black holes lead us to believe that their very existence lends support to the physical bounds presented in this paper and the relatively large quantum fluctuations of space-time given by Eq. (8). By the same token, detection of the space-time foam [Eq. (8)] will be an indirect verification of Hawking black hole evaporation. From our perspective, black hole physics is intimately related to space-time foam physics.

Finally, a comment on the main difference between our approach and that of Lloyd on the physical limits to computation in Ref. [1] is in order. Lloyd's use of the Heisenberg energy-time uncertainty principle to find ν is tantamount to putting $T \sim t$ in Wigner's inequality [Eq. (2)]. In other words, while we have introduced two time scales T and t, Lloyd has introduced only t. But as the case of black holes shows, these two time scales are not the same in general. For a 1-kg black hole, according to Lloyd [1], $\nu \sim 10^{51}$ sec^{-1} and $I \sim 10^{16}$ bits; but according to us, $\nu \sim 10^{35}$ sec^{-1} and $I \sim 10^{16}$ bits [16]. We conclude that we disagree with the limits given in Ref. [1].

To summarize, we have shown that the laws of quantum mechanics and gravitation, which govern the quantum fluctuations of space-time, also set physical bounds on computation and on the precision of clocks. Power limits a simple computer's speed of computation ν and its memory space I. Their product obeys the universal bound given by $I\nu^2 \lesssim t_P^{-2} \sim 10^{86}$ sec^{-2}. This bound is realized for black holes. The same physics underlies the holographic principle. We have also argued that the quantum fluctuations of space-time are actually much larger than what the folkfore suggests. We urge the experimentalists, especially those in the gravitational-wave interferometer field [5,7,10], to strive to detect them.

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[15] The no-hair theorem for black holes already supports such a viewpoint. In some sense, therefore, it is not surprising that black holes belong to the category of simple clocks and simple computers. Note again the qualification "simple."
[16] One can also mention that, for the bounds on I, Lloyd models his computers as systems of ideal gas but switches to the known expression of entropy for the case of black holes. This treatment of the bounds on I is not uniform and its self-consistency has yet to be shown.