Deflection of Spacecraft Trajectories as a New Test of General Relativity

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We derive a simple formula which gives the general relativistic deflection of a spacecraft, idealized as a point mass, for all values of the asymptotic speed V_∞ ($0 \leq V_\infty \leq 1$). Using this formula we suggest a new test of general relativity (GR) which can be carried out during a proposed interstellar mission that involves a close pass of the Sun. We show that, with foreseeable improvements in spacecraft tracking sensitivity, the deflection of a spacecraft's trajectory in the gravitational field of the Sun could provide a new test of GR.

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Recent decades have witnessed significant improvements in the sensitivities of the "classic" tests of general relativity (GR): the gravitational redshift, light deflection, the Shapiro time delay, and the precession of the perihelion of Mercury [1–4]. To a great extent these improvements have been made possible by the introduction of new technologies, such as the Mössbauer effect and atomic clocks in the case of the redshift. Other related tests of GR, including lunar laser ranging [2] and tests of both the weak equivalence principle and the gravitational inverse-square law [5], have also benefited from the use of new technology. In this paper we propose a new test of GR, namely, the deflection of a spacecraft trajectory in the Sun's gravitational field. As with other tests of GR, this test also capitalizes on recent and foreseeable improvements in technology, specifically in both spacecraft tracking and in drag-free systems. Our attention will focus specifically on the small Interstellar Probe mission [6] which is the only mission currently under consideration capable of carrying out a GR test of the type we are proposing.

We begin by assuming that a photon or a spacecraft (idealized as a massive particle) approaches a gravitating body from a very great distance (starting with velocity $\mathbf{V}_\infty^-$) and is deflected by gravity. It recedes to a great distance with final velocity $\mathbf{V}_\infty^+$ (see Fig. 1). Let $\phi(r)$ be the angle, measured positively (by right-hand rule) from the inertial direction $\hat{\mathbf{j}}$ to the position vector direction, $\hat{\mathbf{e}}_r$, as shown in Fig. 1. We then define $\phi(r \to \infty) \equiv \phi_\infty$, and also note that $\phi(r_p) = -\pi/2$, where r_p is the distance of closest approach as shown in the figure. From the symmetry between the approach asymptote and the departure asymptote, we can express the total deflection due to gravity, $\Delta\phi_{\mathrm{def}}$, as

$$\Delta\phi_{\mathrm{def}} = 2[\phi_\infty - \phi(r_p)] - \pi\,. \tag{1}$$

We can now make use of the quadrature integral given by Weinberg [1],

$$\phi = \pm \int \frac{A^{1/2}(r)\,dr}{r^2[\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2}]^{1/2}}\,, \tag{2}$$

where $A(r)$ and $B(r)$ can be expanded in terms of the constants β and γ of the parametrized post-Newtonian (PPN) metric [2],

$$A(r) = 1 + 2\gamma\frac{GM}{r} + \ldots, \tag{3}$$

$$B(r) = 1 - 2\frac{GM}{r} + 2\frac{G^2M^2}{r^2}(\beta - \gamma) + \ldots. \tag{4}$$

In Eqs. (3) and (4), G is the Newtonian gravitational constant, and E and J are constants given by

$$E = 1 - V_\infty^2, \tag{5}$$

$$J = r_p[1/B(r_p) - 1 + V_\infty^2]^{1/2}. \tag{6}$$

Upon substituting Eqs. (3)–(5) into Eq. (2), we obtain, to order G^2,

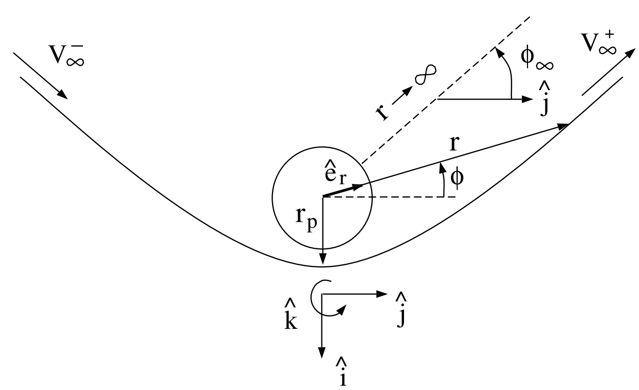

FIG. 1. Deflection of a spacecraft trajectory in a gravity field. The spacecraft approaches with asymptotic velocity $\mathbf{V}_\infty^-$, passes through periapsis at distance r_p, and leaves with asymptotic velocity $\mathbf{V}_\infty^+$. The spacecraft coordinates are given by the radial distance r from the center of the attracting body and the angle ϕ with respect to the inertial direction $\hat{\mathbf{j}}$.

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$$\phi_\infty - \phi(r) = \int_r^\infty \frac{(\frac{1}{r} + \frac{\gamma GM}{r^2})\, dr}{\{\frac{V_\infty^2}{J^2} r^2 + \frac{2GM}{J^2} r - 1 + \frac{2G^2M^2}{J^2}(2 - \beta + \gamma)\}^{1/2}}. \tag{7}$$

After performing the integration in Eq. (7) and substituting for J from Eq. (6), we can (after some tedious algebra) express Eq. (1) in the form

$$\Delta\phi_{\mathrm{def}} = 2\gamma\epsilon\left(\frac{x}{2 + x}\right)^{1/2} + \epsilon\pi\,\frac{(2 + 2\gamma - \beta)}{(2 + x)} + 2\left[1 + \epsilon\,\frac{(2 + 2\gamma - \beta)}{(2 + x)}\right]\sin^{-1}\left(\frac{1}{1 + x}\right), \tag{8}$$

$$\epsilon \equiv GM/r_p \equiv \mu/r_p; \qquad x \equiv V_\infty^2/\epsilon, \tag{9}$$

where we have neglected terms of order ϵ^2 and higher. We recognize in Eq. (8) the classical nonrelativistic deflection of a spacecraft trajectory, $\Delta\phi_{\mathrm{NR}}$,

$$\Delta\phi_{\mathrm{NR}} \equiv 2\sin^{-1}\left(\frac{1}{1 + x}\right). \tag{10}$$

Equation (10) gives the total turn angle of the vector $\mathbf{V}_\infty$ (i.e., the angle between the approach velocity, $\mathbf{V}_\infty^-$, and the departure velocity, $\mathbf{V}_\infty^+$) based on Newton's law of gravity. If we substitute $V_\infty = 1$, or $x = 1/\epsilon$, into Eq. (10), we obtain the deflection of light predicted by Newtonian physics,

$$\Delta\phi_{\mathrm{NR}}\left(\frac{1}{\epsilon}\right) \cong 2\epsilon\,. \tag{11}$$

Similarly, setting $V_\infty = 1$ in Eq. (8) yields

$$\Delta\phi_{\mathrm{def}}\left(\frac{1}{\epsilon}\right) = 2\epsilon(1 + \gamma) = \frac{4GM}{r_p}\left(\frac{1 + \gamma}{2}\right), \tag{12}$$

where terms $O(\epsilon^2)$ and higher have been dropped. Equation (12) yields Einstein's formula for the deflection of light when γ is set to unity: twice the value given by Eq. (11).

We wish to obtain a formula that conveniently compares the general relativistic effect on spacecraft deflection to light deflection. One way to proceed is to define a quantity $\Delta\bar{\phi}$ obtained by subtracting the (often large) angle $\Delta\phi_{\mathrm{NR}}$ in Eq. (10) from the expression in Eq. (8), and to then normalize the result (i.e., divide) by $2\epsilon(1 + \gamma)$:

$$\begin{aligned}\Delta\bar{\phi} &\equiv (\Delta\phi_{\mathrm{def}} - \Delta\phi_{\mathrm{NR}})/[2\epsilon(1 + \gamma)] \\ &= \left(\frac{\gamma}{1 + \gamma}\right)\left(\frac{x}{2 + x}\right)^{1/2} \\ &\quad + \left(\frac{1}{1 + \gamma}\right)\frac{(2 + 2\gamma - \beta)}{(2 + x)}\cos^{-1}\left(\frac{-1}{1 + x}\right).\end{aligned} \tag{13}$$

For Einstein's theory, $\beta = \gamma = 1$ and Eq. (13) becomes

$$\Delta\bar{\phi}_E = \frac{1}{2}\left(\frac{x}{2 + x}\right)^{1/2} + \frac{(3/2)}{(2 + x)}\cos^{-1}\left(\frac{-1}{1 + x}\right). \tag{14}$$

The function $\Delta\bar{\phi}_E$ is plotted in Fig. 2. We note from Eq. (14) that, when $V_\infty = 1$, then $\Delta\,\bar{\phi}_E \cong 0.5$, as expected: the ratio of the purely relativistic bending of light divided by the total bending of light (including the Newtonian bending) is one half. In contrast, for a parabolic trajectory $V_\infty = 0$ (i.e., $x = 0$), and $\Delta\bar{\phi}_E(0) = 3\pi/4 = 2.36$. When $x = 1$, then $V_\infty = \sqrt{GM/r_p}$ which is the circular speed at a radius r_p. Thus the x variable is conveniently scaled in terms of "circular speeds" at r_p. For $x = 1$, $\Delta\bar{\phi}_E = 1.34$. We see that there are many cases where the relativistic deflection of a spacecraft trajectory is greater than the deflection of light for the same periapsis distance, r_p. The question to be answered is whether an experiment can be devised to measure this effect.

As noted previously, Mewaldt *et al.* [6] have proposed the Small Interstellar Probe mission which would cross the solar wind termination shock and heliopause and penetrate into nearby interstellar space. To accomplish its scientific objectives, the probe must attain $V_\infty \approx 1.3 \times 10^{-4}$. To achieve this speed a number of gravity assist scenarios are suggested [6], most of which involve a final close flyby of the Sun at 4 solar radii. At perihelion a maneuver is performed to change the speed of the spacecraft by several km/s in order to send the probe off on its hyperbolic trajectory. The Interstellar Probe mission presents an ideal trajectory to observe the relativistic deflection, provided that the effects of nongravitational forces and the Newtonian deflection can be accounted for. We will use this mission for our numerical example.

Let us first consider how accurately the relevant parameters must be known in order to discriminate between the

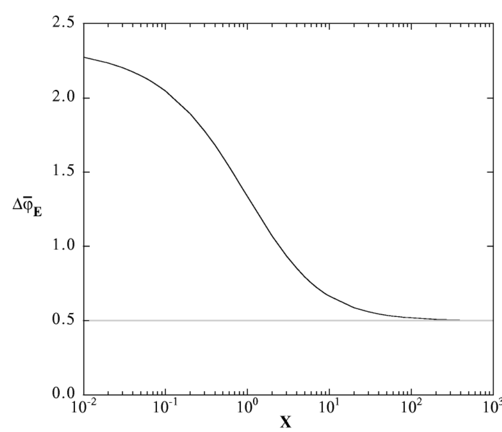

FIG. 2. Plot of the function $\Delta\bar{\phi}_E$ in Eq. (14). As discussed in the text, $\Delta\bar{\phi}_E$ gives the scaled contribution of GR to the deflection, plotted versus the scaled speed x. For an incident light ray $\Delta\bar{\phi}_E = 1/2$.

relativistic and Newtonian deflections. By using $r_p = 4 \times 6.96 \times 10^5$ km $= 2.78 \times 10^6$ km and $GM = 1.475$ km, we find $\epsilon = 5.30 \times 10^{-7}$ and $x = 3.02 \times 10^{-2}$. Inserting these values of ϵ and x into Eq. (14), and multiplying by 4ϵ, gives the total general relativistic deflection $\Delta\phi_E = 4.67 \times 10^{-6}$ rad $= 0.963''$. On the other hand, the nonrelativistic Newtonian deflection is, from Eq. (10), $\Delta\phi_{\mathrm{NR}} = 2.66$ rad $= 152°$, which is very large compared to the relativistic deflection. Thus in order to observe the relativistic deflection we must have very precise knowledge of the Newtonian contribution. (Of course in the case of the Interstellar Probe we will only observe the departure asymptote, namely, half the deflections given by $\Delta\phi_E$ and $\Delta\phi_{\mathrm{NR}}$.) We proceed to assess the sensitivity of measuring the relativistic effect, which will be proportional to the knowledge errors in the nonrelativistic effect.

We can view the rotation induced by general relativity on a hyperbolic trajectory as being a shift in the argument of periapsis of the probe trajectory due to the gravitational interaction, analogous to the advance in Mercury's perihelion. Thus, in order to determine if this is a measurable effect, we must devise a series of ideal measurements to estimate the shift in argument of periapsis between perihelion and escape. At perihelion the argument of periapsis is related to the unit vector of the probe (assuming orbit plane coordinates) by the equation

$$\hat{\mathbf{r}}_p = \cos(\omega)\hat{\mathbf{i}} + \sin(\omega)\hat{\mathbf{j}}, \tag{15}$$

where ω is the argument of periapsis (arbitrarily set to zero in Fig. 1) and $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors of our coordinate frame. When the probe is sufficiently far from the Sun on its escape trajectory, its asymptote can similarly be specified by the unit vector

$$\hat{\mathbf{r}}_\infty = \cos(\omega' + \theta_\infty)\hat{\mathbf{i}} + \sin(\omega' + \theta_\infty)\hat{\mathbf{j}}, \tag{16}$$

where ω' is the new (shifted) argument of periapsis, and θ_∞ is the limiting value of the true anomaly of the probe as it escapes from the Sun. In principle, each of these unit vectors can be measured, and the shift in argument of periapsis can be computed by comparing them. Specifically,

$$|\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty| = \sin(\omega' - \omega)\cos\theta_\infty + \cos(\omega' - \omega)\sin\theta_\infty, \tag{17}$$

and we define $\omega' - \omega \equiv \Delta\phi$, which is the quantity we wish to measure. Noting that $\Delta\phi \ll 1$, $\cos\theta_\infty = -1/e$, and $\sin\theta_\infty = \sqrt{e^2 - 1}/e$, where e is the eccentricity, we can solve for $\Delta\phi$ in terms of measurable quantities,

$$\Delta\phi = \sqrt{e^2 - 1} - e|\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty|. \tag{18}$$

We next take the variation (δ) of the measurement Eq. (18) to compute how errors in measuring the eccentricity and the unit vectors contribute to errors in the measured value of $\Delta\phi$:

$$\delta\Delta\phi = \left[\frac{e}{\sqrt{e^2 - 1}} - |\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty|\right]\delta e - e\delta|\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty|. \tag{19}$$

Noting that $|\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty| \cong \sqrt{e^2 - 1}/e$ reduces Eq. (19) to

$$\delta\Delta\phi = \frac{1}{e\sqrt{e^2 - 1}}\delta e - e[(\hat{\mathbf{k}} \times \hat{\mathbf{r}}_p) \cdot \delta\hat{\mathbf{r}}_\infty + (\hat{\mathbf{r}}_\infty \times \hat{\mathbf{k}}) \cdot \delta\hat{\mathbf{r}}_p], \tag{20}$$

which represents the effect of variations in the angular position of the probe at periapsis and at escape. Careful evaluation of each term for a general flyby shows that the expression in square brackets in Eq. (20) can be expressed as

$$e[\cdots] = e\delta|\hat{\mathbf{r}}_p \times \hat{\mathbf{r}}_\infty| = \Delta r_\infty/r_\infty + \Delta r_p/r_p, \tag{21}$$

where Δ denotes errors in distance measured normal to the radius vector. Since the eccentricity is, in turn, a function of specific measurable quantities via the relation $e = [1 + (r_p V_\infty^2/\mu)]$, we have

$$\delta e = (e - 1)[\delta r_p/r_p + 2\delta V_\infty/V_\infty - \delta\mu/\mu]. \tag{22}$$

where δr_p denotes variations along the radius vector. If we combine the previous results and assume that the different measurements are uncorrelated, then the overall Gaussian uncertainty in $\Delta\phi$ is

$$\sigma_{\Delta\phi}^2 = \frac{1}{e^2}\frac{e - 1}{e + 1}\left[\left(\frac{\sigma_{r_p}}{r_p}\right)^2 + 4\left(\frac{\sigma_{V_\infty}}{V_\infty}\right)^2 + \left(\frac{\sigma_\mu}{\mu}\right)^2\right] + \left[\left(\frac{\sigma_{\Delta r_p}}{r_p}\right)^2 + \left(\frac{\sigma_{r_{\Delta\infty}}}{r_\infty}\right)^2\right], \tag{23}$$

where σ denotes the Gaussian standard deviation of the measured quantity.

In general, the uncertainties in the first terms will be negligible compared to the measured uncertainties $\sigma_{\Delta r_p}$ and $\sigma_{\Delta r_\infty}$. Additionally, at escape the probe unit vector direction can be measured extremely accurately using established differential Very Long Baseline Interferometry (ΔVLBI) techniques [7]. This leaves the down-track measurement of the probe position at perihelion as the dominant error source, so that $\sigma_{\Delta\phi} \approx \sigma_{\Delta r_p}/r_p$. Current navigation capability can reduce $\sigma_{\Delta r_p}$ to the order of 1–10 km [8]. Taking $\sigma_{\Delta r_p} = 1$ km for our numerical example (where e is computed to be 1.03), we find that $\sigma_{\Delta\phi} = 3.6 \times 10^{-7}$ rad which, by comparison to half the deflection angle $\Delta\phi_E$, represents an error of 16%. If this measurement uncertainty were reduced to the order of 10 m, then the contribution of the general relativistic PPN parameters β and γ could be found to three significant figures. Measurement uncertainties of this order imply Earth-based measurement accuracies on the order of 0.1 nanoradians (nrad). Based on operationally demonstrated measurements of the Deep Space Network's ΔVLBI system, its estimated accuracy at present is of order

5 nrad. Observations of natural radio sources made with the ΔVLBI measurement technique have demonstrated accuracies of 0.8 nrad, and the fundamental limit on such measurements is of order 0.01 nrad [9]. This is substantially better than the 0.1 nrad accuracy required to measure β and γ at the $\sim 10^{-3}$ level. The feasibility of developing this technology to the levels of accuracy needed for our proposed experiment and for near-Sun observations is considered in Ref. [10]. We can presume that the increases in accuracy of ΔVLBI will be accompanied by corresponding improvements in the infrastructure needed to support these measurements, which would likely be developed in concert with improved ΔVLBI technology. In order to discriminate the effect of the Sun's quadrupole moment J_2 (which will introduce an error of about 1% in $\sigma_{\Delta\phi}$), a highly inclined orbit is necessary [8]. To disentangle the parameters β and γ in Eq. (13), we rely on recent experiments [11] which determine γ to a precision of $\sim 10^{-3}$.

In order to extract the general relativistic contribution to the spacecraft's trajectory it will be necessary to deal with perturbing nongravitational forces. For a typical spacecraft these forces arise from radiation pressure, solar wind, interplanetary dust, atmospheric drag, magnetic fields, propellant leakage, and spacecraft radiation [4,12]. Such perturbations can be sidestepped by employing a drag-free spacecraft which uses small thrusters to null out the nongravitational forces. Fortunately the necessary drag-free technology is already under development, since it is a prerequisite for the ongoing Gravity Probe B mission [13], as well as for the proposed STEP [14] and Galileo Galilei [15] missions.

We conclude that a variation of the proposed Interstellar Probe mission could measure the general relativistic PPN parameters β and γ to within 16% with current technology. If the perihelion of the spacecraft is determined to within 10 m, then this uncertainty decreases to 0.2%. This level of accuracy should be achievable in the foreseeable future [7,9,10]. This would lead to a new and independent check of GR by utilizing the deflection of spacecraft trajectories.

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