Exclusion Statistics in a Trapped Two-Dimensional Bose Gas

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We study the statistical mechanics of a two-dimensional Bose gas with a repulsive delta-function interaction, using a mean-field approximation. By a direct counting of states we establish that this model obeys exclusion statistics and is equivalent to an ideal exclusion-statistics gas. We also show that this result is consistent with a full quantum-mechanical treatment of a quasi-two-dimensional system.

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The concept of fractional exclusion statistics (FES) proposed by Haldane in 1991 [1] has proved to be a useful concept to describe the statistical mechanics and thermodynamics of certain one-dimensional models [2–5].

The basic idea of FES is that adding a number of particles, ΔN, to a system blocks Δd of the states available for the next particle according to the linear relation $\Delta d = -g\Delta N$. Intuitively this corresponds to a repulsion between the particles, but only very special types of interactions give rise to this type of exclusion of single particle states. In fact, all established examples of FES are in one dimension (or are effectively one-dimensional, like charged particles restricted to the lowest Landau level by a strong magnetic field [4–6]). Thus, the observation [7,8] that, in a Thomas-Fermi approximation, a two-dimensional Fermi or Bose gas with short range repulsive interactions has the same energy and number density as an ideal FES gas (treated in the same approximation) deserves further study. It is not obvious why this kind of interaction should give rise to the exclusion of states in the sense of Haldane, and it is the purpose of this Letter to provide a statistical mechanics derivation.

We start from the two-dimensional Hamiltonian

$$H = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + V(\vec{r}_i)\right) + \frac{\pi\hbar^2}{m}\, g \sum_{i<j}^{N} \delta^2(\vec{r}_i - \vec{r}_j) \quad (1)$$

that has been used as a model for atoms in Bose condensation experiments using highly asymmetric traps [9,10]. The particular form of the delta function is chosen as to reproduce the s-wave scattering phase shifts in three dimensions, and the dimensionless coupling g is given by

$$g = 2\sqrt{\frac{2}{\pi}}\,\frac{a}{l_z}\,, \quad (2)$$

where a is the (three-dimensional) scattering length and l_z is the out of plane extension of the asymmetric trap, which is harmonic in the transverse direction with a frequency $\omega = \hbar/ml_z^2$ [8,9]. (Our definition of g differs from that in [8] for reasons that become clear below.) We assume that the temperature is sufficiently high above the transition temperature that the only relevant mean field is the density, n, and that the fluctuations are small enough to be ignored. We return to these issues at the end of the paper.

Before we analyze the statistical mechanics of (1), we give a simple thermodynamic argument as to why, at the mean-field level, we expect exclusion statistics. For simplicity we consider the case with a constant external potential V, so that the density, n, is also constant. In a mean-field approximation, and for a fixed number of particles, the interaction energy term in (1) just amounts to a constant shift of the energy density,

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\text{free Bos.}} + \frac{1}{2}\, kT\lambda_T^2 n^2 g \\ &= \frac{kT}{\lambda_T^2}\sum_{p=1}^{\infty}\frac{B_{p-1}}{p!}\,(\lambda_T^2 n)^p + \frac{1}{2}\, kT\lambda_T^2 n^2 g\,, \quad (3)\end{aligned}$$

where $kT = 1/\beta$ is the temperature and $\lambda_T = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wavelength. We have used that for a free Bose gas in two dimensions the pressure equals the energy density and substituted the pertinent virial expansion as expressed in the Bernoulli numbers B_n [11]. This expression is consistent with the system being an ideal FES gas in two dimensions which is known to have a pressure equal to the energy density and which differs from a free Bose gas only by a shift $\frac{1}{2}g\lambda_T^2$ in the second virial coefficient [12].

Let us now consider the statistical mechanics of (1) and assume that the potential V is slowly varying compared with the thermal wavelength, λ_T. We then divide the system into cells of area b^2, where $\lambda_T \ll b \ll |\vec{\nabla} V/V|$, and study the statistical mechanics in each cell. In a mean-field approximation, the one-body Hamiltonian in the cell ℓ becomes

$$H_\ell = \frac{p^2}{2m} + V(\vec{r}_\ell) + \frac{2\pi\hbar^2}{m}\, gn(\vec{r}_\ell)\,, \quad (4)$$

where we approximate the potential in the cell with the constant $V(\vec{r}_\ell)$ with $\vec{r}_\ell$ the position of the center of the cell. Also $n(\vec{r}_\ell) = N_\ell/b^2$ is the mean number density in the cell ℓ, with $N_\ell \gg 1$ the corresponding average number of particles. In going from (1) to (4) it is important to correctly incorporate the effect of Bose statistics [13]. There is an extra factor of 2 in the interaction term in the Heisenberg equation for the quantum field operator as compared to the similar looking Gross-Pitaevskii equation for a classical Bose field. This also implies an extra factor of 2 in the mean-field one-body Hamiltonian (4), and here we differ from the treatment in [8].

The total energy E_ℓ is, as usual, not simply the sum of the one particle energies ϵ_i^ℓ of (4) but is given by

$$E_\ell = \sum_i \left[\epsilon_i^\ell - \frac{\pi \hbar^2}{m} g n(\vec{r}_\ell) \right], \tag{5}$$

where the last term compensates for the double counting of the interaction energy.

The number d_ℓ of available one particle quantum states in the box ℓ below some energy ϵ^ℓ, given that there are already N_ℓ particles present, follows from (4)

$$d_\ell = \frac{mb^2}{2\pi\hbar^2}\epsilon_{\mathrm{kin}}^\ell = \frac{mb^2}{2\pi\hbar^2}[\epsilon^\ell - V(\vec{r}_\ell)] - gN_\ell, \tag{6}$$

with $\epsilon_{\mathrm{kin}}^\ell$ the kinetic energy. Note that $g = 1$ corresponds to free fermions, and for general g, this relation immediately hints at exclusion statistics; the number of states in the box decreases linearly with N_ℓ. For a Hamiltonian of the form (4), this is true only in two dimensions.

We can, however, still not conclude that our system is identical to an ideal FES gas. Haldane's original definition of FES was for systems with a finite dimensional single particle Hilbert space, but it was later generalized as to include ideal gases, and the corresponding distribution functions were derived [3,14]. We now demonstrate that the box Hamiltonian H_ℓ in (4) indeed describes an ideal FES gas.

Because of (5), a microstate in the box can be labeled by a set of integers $0 \le k_1 \le k_2 \cdots \le k_{N_\ell}$, with the corresponding energy,

$$E_\ell = \sum_{i=1}^{N_\ell}\left(V(\vec{r}_\ell) + \frac{2\pi\hbar^2}{mb^2}\left[k_i + \frac{g}{2}N_\ell\right]\right). \tag{7}$$

Note that the integers k_i are *not* proportional to the momenta; they are proportional to the kinetic energy and correspond to the number of states below the energy ϵ_i^ℓ. Of course, the real energy eigenvalues are not equally spaced, but we assume that the box is large enough for this effect to be negligible.

Next we introduce the quantities $\tilde{k}_i$ by

$$\tilde{k}_i = k_i + g\sum_j \theta(\tilde{k}_i - \tilde{k}_j), \tag{8}$$

with $\theta(x)$ being the step function, and rewrite the energy as a sum of "pseudoenergies," $\tilde{\epsilon}_i^\ell$,

$$E_\ell = \sum_{i=1}^{N_\ell} \tilde{\epsilon}_i^\ell, \qquad \tilde{\epsilon}_i^\ell = V(\vec{r}_\ell) + \frac{2\pi\hbar^2}{mb^2}\tilde{k}_i. \tag{9}$$

The exclusion properties of this system are now manifest since (8) implies that the pseudoenergies must satisfy

$$\epsilon_{i+1}^\ell \ge \epsilon_i^\ell + \frac{2\pi}{mb^2}\hbar^2 g. \tag{10}$$

It should be noted that the relation (6) holds only because the kinetic energy and the number density scale as the same power of the cell size b. This is true for the present case of particles with quadratic dispersion in two dimensions, but also for particles in one dimension with linear dispersion, which is the case for anyons in the lowest Landau level, or equivalently, chiral particles on a circle with an N^2-type interaction [6]. In fact, the two models studied in this paper and in [6] can be exactly mapped onto each other by identifying the $\tilde{k}_i$ in (8) with the "pseudomomenta" introduced in [6].

The interaction strength g enters only in the combination $\alpha = g\hbar^2$. Recently it was shown that by taking the limit $g \to \infty$, $\hbar \to 0$ with α fixed, α can be interpreted as a *classical* exclusion statistics parameter [15]. Using the Thomas-Fermi approximation for the system (1), and taking the high T limit, it is easy to show that the density is given by $n(\vec{r}) = n_B(\vec{r})[1 + \alpha\frac{2\pi}{kT}n_B(\vec{r})]^{-1}$, where $n_B(\vec{r})$ is the density of a noninteracting Stefan-Boltzmann gas in the same potential $V(\vec{r})$. Note that all $\hbar$ dependence is gone and that the classical density is lowered because of a *classical* statistics effect.

The formal proof that the exclusion property (10) corresponds to an ideal FES gas as defined in [14] is a straightforward modification of the one given in [16] for a multispecies system in the fermionic representation: Going to a continuum description with "momenta" p_i and pseudomomenta $\tilde{p}_i$ defined as

$$p_i = \left(\frac{2\pi\hbar}{b}\right)^2 k_i; \qquad \tilde{p}_i = \left(\frac{2\pi\hbar}{b}\right)^2 \tilde{k}_i, \tag{11}$$

replacing the sums over k and $\tilde{k}$ by integrals in the usual way, and denoting the corresponding particle densities in momentum space by $\nu(p)$ and $\rho(\tilde{p})$, respectively, one finds the continuum version of Eq. (8),

$$\tilde{p} = p + g\int d\tilde{p}'\,\rho(\tilde{p}')\theta(\tilde{p} - \tilde{p}'). \tag{12}$$

Furthermore one has to demand conservation of the number of particles when changing variables from p to $\tilde{p}$, i.e., $\nu(p)dp = \rho(\tilde{p})d\tilde{p}$. Combining this with Eq. (12) gives

$$\nu(p) = \frac{\rho(\tilde{p})}{1 - g\rho(\tilde{p})}. \tag{13}$$

Inserting this into the standard expression for the bosonic nonequilibrium entropy,

$$S = -k\left(\frac{b}{2\pi\hbar}\right)^2 \int dp\,[\nu \ln\nu - (1 + \nu)\ln(1 + \nu)], \tag{14}$$

exactly reproduces the entropy of an ideal FES gas [3,14],

$$S = -k\left(\frac{b}{2\pi\hbar}\right)^2 \int d\tilde{p} \times \{\rho \ln\rho - [(1 - (g - 1)\rho]\ln[1 - (g - 1)\rho] + (1 - g\rho)\ln(1 - g\rho)\}, \tag{15}$$

from which all thermodynamics follows.

Since we have explicitly ignored both the possibility of a quantum condensate and of pairing fields other than the density, the results of this paper (and those of Ref. [8]) can not be used for temperatures below or in the vicinity of the Bose condensation transition T_c. This is true irrespective of whether this transition is of the Kosterlitz-Thouless type or not [17]. Rather our results should be relevant in a temperature regime where the exclusion statistics, due to the repulsive interaction, corresponds to a small correction to the ideal Bose gas. It is an interesting open question whether the quasiparticles above a two-dimensional Bose condensate can also be described using exclusion statistics. To answer this question one would analyze the corresponding statistical mechanics in a more sophisticated mean-field approximation that includes effects of phase coherence and pairing mean fields [13].

So far our analysis has been entirely in the context of mean-field approximations. It is an interesting question whether the full quantum problem of a two-dimensional gas with a delta function interaction also allows a description in terms of exclusion statistics in some range of temperatures. Although the interaction naively does not involve any dimensionful parameter, it is known that a pure delta function interaction gives rise to short distance singularities and requires a renormalization of the interaction strength which introduces a renormalization scale and thus breaks scale invariance [18]. In quantum mechanics, the second virial coefficient is related to the scattering phase shifts by the Beth-Uhlenbeck formula [19]. The s-wave scattering phase shifts in the quasi-2d problem can be calculated as

$$\cot\delta_0 = -\frac{4}{\pi g} + \frac{2}{\pi}\ln\left(\frac{p l_z^{\mathrm{eff}}}{\hbar}\right), \tag{16}$$

where p is the relative momentum in the two-body scattering process and g is related to the (three-dimensional) scattering length by (2). To derive (16) one can either employ the Greens function method used in [17] or directly match the wave functions as explained in [20]. Note that the phase shift does depend on the momentum p via the renormalization scale l_z^{eff}, which up to a numerical factor equals the transverse extent l_z of the quasi-2d system. In a strict two-dimensional system with a delta function interaction, there is always a single bound state, which, however, is not present in the quasi-2d case. The Beth-Uhlenbeck formula then gives the following shift in the pressure due to interactions [21]:

$$\Delta P = -(n\lambda_T)^2 \frac{2kT}{\pi} \int_0^\infty dp\, e^{-\frac{p^2}{mkT}} \frac{d\delta_0}{dp}. \tag{17}$$

In general, the integral in (17) is a function of l_z/λ_T, but in the relevant parameter range, $a \ll l_z \ll \lambda_T$, it is approximately constant and we get to leading order in g,

$$\Delta P = \frac{1}{2} kT(n\lambda_T)^2 g, \tag{18}$$

in perfect agreement with (3). We can thus conclude that for temperatures and couplings in the range specified above, a full quantum mechanical treatment is consistent with the mean-field approximation used earlier. It is an open question whether there are any corrections to higher virial coefficients.

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