Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED

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We have calculated the energy levels of the hydrogen atom as well as the Lamb shift within the noncommutative quantum electrodynamics theory. The results show deviations from the usual QED both on the classical and the quantum levels. On both levels, the deviations depend on the parameter of space/space noncommutativity.

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I. Introduction.—Recently, remotivated by string theory arguments, noncommutative spaces (Moyal plane) have been studied extensively. The noncommutative space can be realized by the coordinate operators satisfying

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\,, \tag{1.1}$$

where $\hat{x}$ are the coordinate operators and $\theta_{\mu\nu}$ is the noncommutativity parameter and is of dimension of $(\text{length})^2$; for a review on the string theory side, see [1]. The action for field theories on noncommutative spaces is then obtained using the Weyl-Moyal correspondence [2–4], according to which, in order to find the noncommutative action, the usual product of fields should be replaced by the star product

$$(f * g)(x) = \exp\left(\frac{i}{2}\,\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right)f(x)g(y)|_{x=y}\,, \tag{1.2}$$

where f and g are two arbitrary infinitely differentiable functions on R^{3+1}. Performing explicit loop calculations, for $\theta^{0i} = 0$ cases (noncommutative space), it has been shown that noncommutative ϕ^4 theory up to two loops [3,5] and noncommutative QED up to one loop [4,6,7] are renormalizable. For noncommutative space-time ($\theta^{0i} \neq 0$) it has been shown that the theory is not unitary and hence, as a field theory, it is not appealing [8].

Apart from the field theory interests which are more academic, we are more interested in some possible phenomenological consequences of noncommutativity in space. Some of those results, all from the field theory point of view, have been addressed in [4,9–11]. However, perhaps a better starting point is to study quantum mechanics (QM) on such noncommutative spaces. To develop the noncommutative QM formulation we need to introduce a Hamiltonian which governs the time evolution of the system. We should also specify the phase space and, of course, the Hilbert space on which these operators act. As for the phase space, inferred from the string theory [12,13], we choose

$$\begin{aligned}[\hat{x}_i, \hat{x}_j] &= i\theta_{ij}\,,\\ [\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{ij}\,,\\ [\hat{p}_i, \hat{p}_j] &= 0\,.\end{aligned} \tag{1.3}$$

The Hilbert space can consistently be taken to be exactly the same as the Hilbert space of the corresponding commutative system. This assumption for the Hilbert space is directly induced from the nonrelativistic limit of the related noncommutative field theory, and one can really argue that it satisfies all the needed properties of a physical Hilbert space. The only nontrivial part of such a formulation is to give the Hamiltonian. Once we have done it, the dynamical equation for the state $|\psi\rangle$ is the usual Schrödinger equation, i.e., $H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$.

In this Letter we focus on the hydrogen atom and, using the nonrelativistic limit of noncommutative QED results, we propose the Hamiltonian describing the noncommutative hydrogen atom. Given the Hamiltonian and assuming that the noncommutativity parameter (θ_{ij}) is small, we study the spectrum of hydrogen atom. We show that because of noncommutativity, even at field theory tree level, we have some corrections to the Lamb shift ($2P_{1/2} \to 2S_{1/2}$ transition). Since the noncommutativity in space violates rotational symmetry, our Lamb shift corrections have a preferred direction, and hence we call them "polarized Lamb shift." We also consider further corrections to Lamb shift originating from the loop contributions in noncommutative QED. In this way we will find some upper bound for θ. In addition, we study the Stark and Zeeman effects for the noncommutative hydrogen atom.

II. Formulation of the noncommutative Hamiltonian.—To start with, we propose the following Hamiltonian for the noncommutative hydrogen atom. Of course, we shall verify our proposal by a noncommutative QED calculation:

$$H = \frac{\hat{p}\hat{p}}{2m} + V(\hat{x})\,, \tag{2.1}$$

where the Coulomb potential in terms of the noncommutative coordinates $\hat{x}$ is

$$V(r) = -\frac{Ze^2}{\sqrt{\hat{x}\hat{x}}}\,, \tag{2.2}$$

with $\hat{p}$ and $\hat{x}$ satisfying (1.3).

Now, we note that there is a new coordinate system,

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$$x_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j, \qquad p_i = \hat{p}_i, \tag{2.3}$$

where the new variables satisfy the usual canonical commutation relations:

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}, \quad [p_i, p_j] = 0. \tag{2.4}$$

So, if in the Hamiltonian we change the variables $\hat{x}_i$, $\hat{p}_i$ to x_i, p_i, the Coulomb potential becomes

$$\begin{aligned} V(r) &= -\frac{Ze^2}{\sqrt{(x_i - \theta_{ij}p_j/2\hbar)(x_i - \theta_{ik}p_k/2\hbar)}} \\ &= -\frac{Ze^2}{r} - Ze^2\frac{x_i\theta_{ij}p_j}{2\hbar r^3} + O(\theta^2) \\ &= -\frac{Ze^2}{r} - Ze^2\frac{\vec{L}\cdot\vec{\theta}}{4\hbar r^3} + O(\theta^2), \end{aligned} \tag{2.5}$$

where $\theta_i = \epsilon_{ijk}\theta_{jk}$, $\vec{L} = \vec{r}\times\vec{p}$.

As $(\vec{r}\times\vec{p})\cdot\vec{\theta} = -\vec{r}\cdot(\vec{\theta}\times\vec{p})$, it follows that the Coulomb potential can also be written as

$$V(r) = -\frac{Ze^2}{r} - \frac{e}{4\hbar}(\vec{\theta}\times\vec{p})\cdot\left(-\frac{Ze\vec{r}}{r^3}\right) + O(\theta^2). \tag{2.6}$$

The other higher order terms, besides being higher powers in θ, which in its own turn is very small, are also higher powers in momenta.

Our proposal for the Hamiltonian can be justified from field theory calculations. The electron-photon vertex function at tree level in noncommutative QED is [4]

$$\Gamma_\mu = e^{(i/2\hbar^2)p\times p'}\gamma_\mu = e^{-(i/2\hbar^2)p\tilde{q}}\gamma_\mu, \tag{2.7}$$

where p and p' are the incoming and outgoing electron momenta, respectively, q_μ is the photon momentum, $p' - p = q$, and

$$p\times p' = p_i\theta^{ij}p'_j, \qquad \tilde{q}^i = \theta^{ji}q_j.$$

Expanding the exponential in powers of θ and keeping only the first two terms, it appears that the second term will give rise to an *electric dipole moment* [14], which couples to an external electric field E as $-\langle\vec{P}\rangle\cdot\vec{E}$ where

$$\langle P_i\rangle = -\frac{1}{2\hbar}e\tilde{p}_i = \frac{1}{2\hbar}e\theta_{ij}p_j. \tag{2.8}$$

This electric dipole moment, as we will see, changes the usual Lamb shift. Actually one can go further and prove that the potential (2.5), for all orders in θ, is expected from the noncommutative QED starting from (2.7). This can be done noting that $f(x_i + \epsilon_i) = e^{\epsilon_i(\partial/\partial x_i)}f(x)$.

Our proposal for the noncommutative hydrogen atom Hamiltonian can be generalized to other systems, i.e., taking the usual Hamiltonian but now being a function of noncommutative coordinates [like (2.1)]. However, our discussion based on noncommutative QED is applicable only when we deal with the "electromagnetic" interaction. In other words, at field theory tree level and in the nonrelativistic limit, the noncommutativity of space is probed through the electric dipole moment of particles, whether fermions or bosons.

In our formulation for noncommutative quantum mechanics, one can still use the usual definition for the probability density, $|\psi|^2$. However, one should be aware that there is no coordinate basis in this case. In our approach, since the noncommutativity parameter, if it is nonzero, should be very small compared to the length scales of the system, one can always treat the noncommutative effects as some perturbations of the commutative counterpart and hence, up to first order in θ, we can use the usual wave functions and probabilities.

III. "Classical" spectrum for hydrogen atom in NC theory.—Using the usual perturbation theory, the leading corrections to the energy levels due to noncommutativity, i.e., first order perturbation and in field theory tree level, are

$$\Delta E_{NC}^{H-\text{atom}} = -\langle nl'jj'_z|\frac{Ze^2}{4\hbar}\frac{\vec{L}\cdot\vec{\theta}}{r^3}|nljj_z\rangle. \tag{3.1}$$

We note that the above expression is very similar to that of the spin-orbit coupling, where $\frac{\theta}{\lambda_e^2}$ is now replacing the spin, $\frac{\vec{S}}{\hbar}$, with λ_e being the electron Compton wave length.

If we put $\theta_3 = \theta$ and the rest of the θ components to zero (which can be done by a rotation or a redefinition of coordinates), then $\vec{L}\cdot\vec{\theta} = L_z\theta$ and, taking into account the fact that

$$\langle ljj_z|L_z|l'jj'_z\rangle = j_z\hbar\left(1 \mp \frac{1}{2l+1}\right)\delta_{ll'}\delta_{j_zj'_z},$$

$$j = l \pm \frac{1}{2},$$

the energy level shift given by (3.1) becomes [15]

$$\Delta E_{NC}^{H-\text{atom}} = -\frac{m_ec^2}{4}(Z\alpha)^4\frac{\theta}{\lambda_e^2}j_z\left(1 \mp \frac{1}{2l+1}\right) \times f_{n,l}\delta_{ll'}\delta_{j_zj'_z} \tag{3.2}$$

for $j = l \pm \frac{1}{2}$ and $f_{n,l} = 1/n^3l(l+\frac{1}{2})(l+1)$.

It is worth noting that in order to find $\langle\frac{1}{r^3}\rangle$, one should integrate over the wave functions from $r = 0$; on the other hand, the approximation we are working in (dropping the terms higher order in θ) is not valid for $r \lesssim \sqrt{\theta}$. However, since the wave function for $l \neq 0$ is zero at $r = 0$, the result (3.2) still holds at this level.

The case of our interest, the $2P_{1/2} \to 2S_{1/2}$ transition (Lamb shift), differs from the usual commutative case in which the shift depends only on the l quantum number and all the corrections are due to the field theory loop effects. The Lamb shift for the noncommutative hydrogen atom, besides the usual loop effects, depends on the j_z

quantum number (only for the $2P_{1/2}$ level, as the levels with $l = 0$ are not affected) and is there, even in the field theory tree level. Hence we call it *polarized Lamb shift*. More precisely, there is a *new* transition channel which is opened because of noncommutativity: $2P_{1/2}^{-1/2} \to 2P_{1/2}^{1/2}$, with the notation $nl_j^{j_z}$ for the energy levels. The usual Lamb shift, $2P_{1/2} \to 2S_{1/2}$, is now split into two parts, $2P_{1/2}^{1/2} \to 2S_{1/2}$ and $2P_{1/2}^{-1/2} \to 2S_{1/2}$, which means that the noncommutativity effects increase the widths and split the Lamb shift line by a factor proportional to θ.

IV. One loop corrections.—In the usual commutative theory, the Lamb shift is believed to come from *loop corrections* to QED. In the usual case, both vertex corrections, in particular, the $g - 2$ factor in the spin-orbit coupling, and the corrections to photon propagator [16] are responsible for the Lamb shift.

A. Noncommutative one loop vertex corrections: According to noncommutative QED one loop results, the electric and magnetic dipole moments of the electron, as a Dirac particle, are [4]

$$\langle \vec{\mu} \rangle = -\frac{e}{2mc}\left(g\vec{S} + \frac{\alpha\gamma_E}{3\pi}\hbar\frac{\vec{\theta}}{\lambda_e^2}\right), \qquad g = 2 + \frac{\alpha}{\pi},$$

$$\langle \vec{P} \rangle = -\frac{e}{4\hbar}(\vec{\theta} \times \vec{p})\left(1 + \frac{3\alpha\gamma_E}{\pi}\right). \tag{4.1}$$

Hence, the *noncommutative* one loop correction to the potential (2.5), originating from *vertex* corrections up to the first order in θ, is

$$V_{NC\ \text{vertex}}^{1\text{Loop}} = -\frac{Ze^2}{4\pi}\gamma_E\alpha\left(3 - \frac{2}{3}\right)\frac{\vec{L}\cdot\vec{\theta}}{\hbar r^3}. \tag{4.2}$$

B. Noncommutative one loop photon propagator corrections: The photon propagator at one loop in the noncommutative QED, for small q, $\tilde{q}$ is given by [6]

$$\Pi^{\mu\nu}(q) = \frac{e^2}{16\pi^2}\left\{\frac{10}{3}(g^{\mu\nu}q^2 - q^\mu q^\nu) \times \left[\ln(q^2\tilde{q}^2) + \frac{2}{25}\frac{q^2}{m^2}\right] + 32\frac{\tilde{q}^\mu\tilde{q}^\nu}{\tilde{q}^4} - \frac{4}{3}\frac{q^2}{\tilde{q}^2}\tilde{q}^\mu\tilde{q}^\nu\right\}, \tag{4.3}$$

where the term proportional to $\frac{2}{25}\frac{q^2}{m^2}$ is the fermionic loop contribution which, because of the cancellation in phase factors coming from noncommutativity, is the same as the usual QED result. From (4.3), by taking only the part of the propagator corresponding to timelike photons and reintroducing $\hbar$, c factors, in the units where the Coulomb potential is $-\frac{Ze^2}{r}$, we obtain

$$V_{\text{prop.}}^{1\text{Loop}}(r) = -Ze^2\alpha\frac{10}{3\hbar}\int d^3q\,\frac{1}{\vec{q}^{\,2}}e^{-i\vec{q}\cdot\vec{r}/\hbar} \times \left[\ln\left(\frac{\vec{q}^{\,2}\tilde{q}^2}{\hbar^4}\right) + \frac{2}{25}\frac{\vec{q}^{\,2}}{m^2}\right]. \tag{4.4}$$

The second term in the integral yields the usual $\delta^3(r)$-type correction to the Coulomb potential. To work out the integral in the first term which is θ dependent, let us assume that only $\theta_{12} = \theta$ is nonzero. If we denote the integral by $I(r,\theta)$, then $\frac{dI}{d\theta} = \frac{1}{2\pi r\theta}$ and thus $I(\theta, r) = \frac{1}{2\pi r}\ln(\theta\Lambda^2)$, where Λ is a cutoff. This can be understood noting that, because of IR/UV mixing [4,6], the Fourier transformation and also (4.3) are valid for $\frac{1}{\Lambda\theta} \lesssim q \lesssim \Lambda$. Putting all these results together, we have

$$V_{\text{prop.}}^{1\text{Loop}}(r) = -\frac{Ze^2}{2\pi r}\frac{10\alpha}{3}\ln(\theta\Lambda^2) - Ze^2\frac{4\alpha}{15}\lambda_e^2\delta^3(r). \tag{4.5}$$

The first term, being proportional to $\frac{1}{r}$, can be understood as the normalization of charge at the one loop level [6]; however, to find the physical value of α (noncommutative QED coupling), one should study the Thomson limit of Compton scattering [17] for the noncommutative case [18]. Summing up all the one loop contributions to the Lamb shift due to noncommutativity, (4.2) and (4.5), we get

$$\Delta E_{NC}^{1\text{Loop}} = -\frac{1}{2\pi}m_e c^2(Z\alpha)^2 \times \left[\frac{5\alpha}{3}\ln(\theta\Lambda^2)\frac{1}{n^2} - \frac{(Z\alpha)^2}{2}\frac{\theta}{\lambda_e^2} \times \gamma_E\alpha\left(3 - \frac{2}{3}\right)\frac{j_z(1 \mp \frac{1}{2l+1})}{n^3 l(l+\frac{1}{2})(l+1)}\right]. \tag{4.6}$$

One can use the data on the Lamb shift to impose some bounds on the value of the noncommutativity parameter, θ. Of course, to do it, we need to consider only the classical (tree level) results, (3.2). Comparing these results, the contribution of (3.2) should be of the order of 10^{-6}–10^{-7} smaller than the usual one loop result, and hence

$$\frac{\theta}{\lambda_e^2} \lesssim 10^{-7}\alpha \quad \text{or} \quad \theta \lesssim (10^4\ \text{GeV})^{-2}.$$

This bound is indeed not a strong one, and one would need some more precise experiments or data. Among other processes, the e^+e^- scattering data can provide a better bound on θ [18].

V. Noncommutative Stark and Zeeman effects.—Stark effect: The potential energy of the atomic electron in an external electric field oriented along the z axis is given, at tree level, by

$$V_{\text{Stark}} = eEz + \frac{e}{4\hbar}(\vec{\theta} \times \vec{p}) \cdot \vec{E} \tag{5.1}$$

(neglecting the motion of the proton).

The change in the hydrogen atom energy level due to noncommutativity [the second term in (5.1)] is

$$\Delta E_{\text{Stark}}^{NC} = \langle nl'jj_z'|\frac{e}{4\hbar}(\vec{\theta} \times \vec{p}) \cdot \vec{E}|nljj_z\rangle. \tag{5.2}$$

Taking into account the fact that $p_i = \frac{m}{i\hbar}[x_i, H_0]$, where H_0 is the unperturbed Hamiltonian, so that $H_0|nljj_z\rangle = E_n|nljj_z\rangle$, the correction to the energy levels becomes

$$\Delta E^{NC}_{\rm Stark} \propto (\vec{\theta} \times \vec{E})_i \langle nl' jj'_z | [x_i, H_0] | nljj_z \rangle = 0, \quad (5.3)$$

meaning that, at tree level, the contribution to the Stark effect due to noncommutativity is zero. We also note that, adding the one loop corrections to electric dipole moment (4.1), the above result will not be changed.

Zeeman effect: The new parts which are added to the usual potential energy of the atom in a magnetic field, due to noncommutativity, are

$$V_{NC\ \rm Zeeman} = \frac{e}{2m_e c} \frac{\alpha \gamma_E m_e^2}{3\pi\hbar} \left(1 - f \frac{m_p}{m_e}\right) \vec{\theta} \cdot \vec{B}, \quad (5.4)$$

where f is a form factor of the order of unity, as the proton is not pointlike. As a result, the noncommutative contribution to the Zeeman effect in the first order of perturbation theory is

$$\Delta E^{NC}_{\rm Zeeman} = \frac{1}{6\pi c\hbar} e \alpha \gamma_E m_e \left(1 - f \frac{m_p}{m_e}\right) \vec{\theta} \cdot \vec{B}. \quad (5.5)$$

VI. Conclusion.—We have presented the results on the classical Coulomb potential within the formulated noncommutative quantum mechanics for the hydrogen atom and have obtained the corrections to the Lamb shift using the noncommutative QED. If there exists any noncommutativity of space-time in nature, as it seems to emerge from different theories and arguments, its implications should appear in physical systems such as the one treated in this Letter. A detailed analysis of the results obtained here, together with the treatment of other fundamental and precisely measured physical processes, will be given in a further communication [18].

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