Comment on "Breakdown of Bohr's Correspondence Principle"

In a recent Letter, Gao [1] studies deep potential wells with attractive tails of the form $-[\hbar^2/(2\mu)](\beta_n)^{n-2}/r^n$. He relates quantum-classical correspondence to the performance of conventional WKB quantization and, for $n > 2$, he finds WKB to be least accurate for near-threshold states with the largest quantum numbers v and more accurate for lower lying states. These results are consistent with long established conceptions of quantum-classical correspondence and cannot be used to conclude that Bohr's correspondence principle breaks down.

Quantum-classical correspondence may be expected in the semiclassical limit, where quantum wavelengths (in the classically allowed regime) and penetration depths (in the classically forbidden regime) become small compared to classical length scales. It is unusual to expect conventional WKB quantization to be accurate in the *anticlassical limit,* where wavelengths or penetration depths become large in comparison with classical length scales.

For a potential tail as above, the classical length scale at energy $E = -\hbar^2\kappa^2/(2\mu)$ is defined by the classical turning point $r_t = \kappa^{-2/n}\beta_n^{1-2/n}$, and the semiclassical and anticlassical limits depend on the ratio a of r_t to the quantum mechanical penetration depth $1/\kappa$, $a = r_t\kappa = (\kappa\beta_n)^{1-2/n}$ [2]. The semiclassical limit is reached for $a \to \infty$, whereas $a \to 0$ corresponds to the anticlassical limit. For $0 < n < 2$ the threshold limit $\kappa \to 0$ is the semiclassical limit, because r_t goes to infinity faster than $1/\kappa$. For $n > 2$, however, $\kappa \to 0$ corresponds to the anticlassical limit, because the quantum mechanical penetration depth $1/\kappa$ becomes larger and larger in comparison with the classical turning point r_t. For $n > 2$ the semiclassical limit is $\kappa\beta_n \to \infty$, so lower states with larger κ are closer to the semiclassical limit. Hence conventional WKB quantization is more accurate for the lower states, whereas it breaks down near threshold, $\kappa \to 0$.

Bohr's correspondence principle is generally formulated for the limit $v \to \infty$. A given potential well with a potential tail as above and a fixed strength parameter β_n can, for $n > 2$, support only a finite number of bound states; the limit $v \to \infty$ cannot be performed. The limit $\kappa \to 0$ is the anticlassical limit, and the correspondence principle is not meant to apply here.

Quantum-classical correspondence is expected in the semiclassical limit $a \to \infty$. For $n > 2$ this implies $\kappa\beta_n \to \infty$ and can be realized by studying a fixed energy (fixed κ) or a range of energies separated from threshold by a finite bound, and taking the limit $\beta_n \to \infty$. The quantum numbers then grow to infinity and conventional WKB quantization becomes increasingly accurate. This is how Bohr's correspondence principle works in the present case.

The progressive deterioration of conventional WKB quantization towards threshold and its breakdown near threshold for potentials with attractive tails vanishing faster than $1/r^2$ have been observed before [2–6]. The question of how to modify the quantization rule near threshold has received convincing numerical [2,3] and analytical [2,4] answers. For $E \to 0$ the appropriately modified quantization condition has the form

$$v = A - B\sqrt{-E} + O(E), \tag{1}$$

where A and B are constants.

Equation (1) holds quite generally for any attractive potential tail vanishing faster than $1/r^2$, as long as there is a region of values of r in the well where WKB wave functions are accurate solutions of the Schrödinger equation. To determine the constant A we need to know the potential in all of the classically allowed region, but the constant B depends only on the tail of the potential beyond the WKB region. For the potential tail above,

$$B = \frac{\sqrt{2\mu}}{\pi\hbar}\,\frac{\beta_n}{(n-2)^{2/(n-2)}}\,\frac{\Gamma(1-\frac{1}{n-2})}{\Gamma(1+\frac{1}{n-2})}\,\sin\!\left(\frac{\pi}{n-2}\right), \tag{2}$$

as derived by Trost *et al.* [2,4]. For $n = 3$, $B = \beta_3\sqrt{2\mu}/\hbar$.

For a deep potential with an attractive tail vanishing faster than $1/r^2$, the breakdown of conventional WKB quantization near threshold is well understood. It is, however, wrong to relate this anticlassical limit $E \to 0$ to Bohr's correspondence principle. The correspondence principle applies in the semiclassical limit, and this statement holds also for $n > 2$.

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