Self-Organized Critical Drainage Networks

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(Received 23 June 2000; revised manuscript received 13 October 2000)

We introduce time-dependent boundary conditions in a model of drainage network evolution based on local erosion rules. The changing boundary conditions prevent the model from becoming stationary; it approaches a state where fluctuations of all sizes occur. The fluctuations in the sizes of the drainage areas show power law behavior with an exponent that differs significantly from that of the static distribution of the drainage areas. Thus, the model exhibits self-organized criticality and proposes a novel concept for predicting fractal properties of drainage networks.

DOI: 10.1103/PhysRevLett.86.2689 PACS numbers: 92.40.Gc, 05.45.Df, 05.65.+b, 47.53.+n

Introduction.—Fractal properties of drainage networks have been investigated in both field studies and modeling; three types of models were established for understanding them.

(i) *Random walk approaches* [1] are based on random flow directions at the nodes of a discrete grid. These models were the first to generate fractal networks with apparently realistic properties, but further studies [2] showed that this coincidence is not as good as it seemed.

(ii) *Simplified landform evolution models*, especially that of Takayasu and Inaoka [3], use local erosion rules where the erosion rate at the nodes of a discrete grid depends on discharge and slope. The simulation starts at an arbitrary, e.g., random surface and ends up at a steady state with quite realistic fractal properties. The evolution of the network looks like a tree growing from the outlet of the basin.

(iii) The concept of *optimal channel networks* was introduced by Rinaldo *et al.* [4]; it is based on local erosion rules, too. In contrast to landform evolution models, only the final, equilibrated states are considered; among them "optimal" networks are computed by minimizing the total energy dissipation in the system. This concept predicts the fractal properties of natural drainage networks well [2], but the question how nature manages to optimize energy expenditure as a result of a landform evolution process is still open.

Most of these approaches contain random components, but assume constant model parameters such as precipitation rates and erodibilities. In contrast, inhomogeneity is essential in many geological phenomena. The influence of random variations in the model parameters has been investigated, and inhomogeneity seems to improve the results with reference to the properties of real drainage networks [5,6].

After the framework of *self-organized criticality* (SOC) was introduced by Bak *et al.* [7], it has been established as an explanation for the occurrence of fractal patterns in various processes. The idea of interpreting drainage networks as a SOC phenomenon is tempting, but there is a fundamental problem: SOC requires a quasisteady state where fluctuations of all sizes occur, and the fractal size distribution of these fluctuations is the spatial fingerprint of SOC systems. In contrast, the approaches discussed above are based on or result in stationary patterns, and their fractal properties are static. Takayasu and Inaoka [3] considered the behavior of their model as a new type of SOC, but interpreted it as a fractal growth process later. Discussion has shown that it is far away from the SOC concept in its original sense [8]. The random-pinning model [6] introduces some kind of time-dependent randomness which prevents the networks from becoming stationary, but a clear link to SOC through a scale-invariant distribution of fluctuations has not been established yet.

A simple erosion model with variable boundary conditions.—We start at an erosion model that is similar to Takayasu and Inaoka's [3] approach. A grid is defined, and precipitation acts uniformly on the nodes. Each node delivers its discharge to that neighbor where the steepest downslope gradient occurs. Hexagonal [3] and quadratic [4,9] grid topologies are common, and we use a quadratic lattice where diagonal flow directions are allowed, too.

Expressions for the erosion rate can be partly derived from hydrodynamic principles; the most common approximation is [3,9]

$$\frac{\partial H_i}{\partial t} \sim -q_i^{\mu} \Delta_i^{\nu}, \tag{1}$$

where H_i is the surface height at the node i, q_i the discharge, and Δ_i the slope (in direction of steepest descent). μ and ν are positive parameters where $\nu \approx 2\mu$. Following Takayasu and Inaoka [3] we choose $\mu = 1$ and $\nu = 2$. In this case, an implicit time discretization of the slope gradients Δ_i in Eq. (1) can be performed if the discharges are once computed. This avoids instabilities that may propagate in the upstream direction, so that we do not need to limit the erosion rate at large discharges and slopes as Takayasu and Inaoka did.

Obviously, the rules of flow routing do not work if the surface has local depressions. They can be considered as lakes and be filled up with water [3,9]. However, they do not play any part in our model in the long term run, so we simply assume that the water vanishes at local minima without any erosive action.

Various boundary conditions can be posed. We first focus on single-outlet networks where one node at the boundary is chosen as the outlet of the basin. At this outlet, a constant erosion rate is applied, while that at the other nodes is determined by the erosion rule [Eq. (1)]. We call this approach "standard model" in the following.

As expected, this model finally leads to a steady state where the erosion rates of all nodes are the same. This is not realistic; nature provides several influences that prevent the earth's surface from reaching equilibrium, such as temporally variable erodibility due to different layers of bedrock, side erosion (meandering), and changing tectonic conditions.

Our approach introduces time-dependent boundary conditions by assuming that the location of the outlet changes through time. After the surface has become stationary, we choose a new outlet at the boundary randomly, so that there are two outlets at this time. The new outlet is eroded at the same predefined rate as the old one was, while erosion at the old one ceases. As a result, the drainage area of the new outlet grows, while that of the old outlet decreases. Finally, the new outlet captures the whole basin, and the old one is no longer active. After the surface has become stationary again, a new outlet is chosen and the procedure is repeated. Although this approach does not represent any of the natural processes mentioned above directly, it can be seen as a simple representation of changes in the environment of a drainage basin.

Results.—The results presented in this section are obtained from a simulation on a 128×128 lattice. In order to avoid effects of the random initial condition, all results were skipped until each node had changed its drainage direction at least 25 times; this was the case after about 1300 stationary networks were computed. Then, 5000 stationary networks were computed and analyzed.

Figure 1 shows a series of 12 consecutive equilibrated networks. Those parts which have changed since the previous steady state are marked black; grey segments have persisted. Obviously, only parts of the network are reorganized as a response to introducing a new outlet; parts of these alterations simply result from changing the flow direction while the valley itself remains. However, more than 90% of the network are reorganized through these 12 steps. During the whole simulation, each site changed its drainage direction at least 144 times, and in the mean 394 times. This shows that—although consecutive steady state networks are not independent—the simulation provides a sufficient statistics which is not biased by persisting structures.

Six different fractal properties are observed in nature, but only three of them are independent [2]. In the following, we use the exponent β of the size distribution of drainage areas, the exponent h of Hack's law, and the length scaling exponent ϕ_L for model validation.

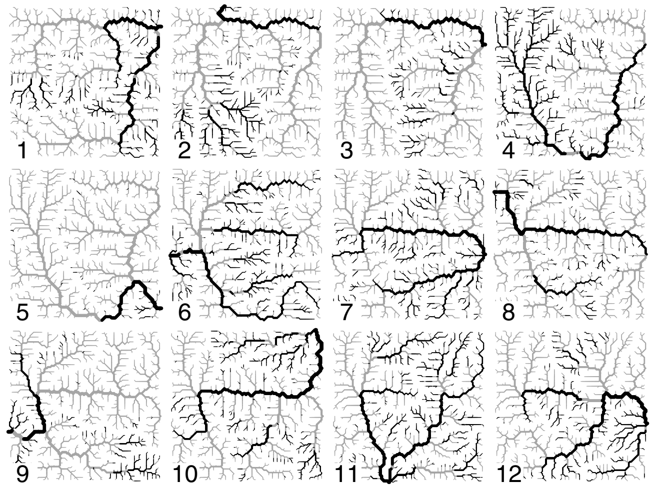

FIG. 1. Series of stationary drainage networks as a result of changing boundary conditions. Only river segments with drainage areas $A \geq 10$ are plotted; the linewidths are proportional to $A^{1/4}$.

Figure 2 shows the cumulative size distribution of the drainage areas, obtained from considering all nodes of the 5000 simulated networks. Except for a finite size effect at large areas, the distribution shows power law behavior for areas larger than 10 nodes: $P(A) \sim A^{-\beta}$, where $P(A)$ is the probability that the drainage area of a point exceeds A. The exponent $\beta = 0.46$ is at the upper edge of the observed range $\beta \in [0.41, 0.46]$ [2,10].

For comparison, the distribution obtained from the standard model is included in the plot. A total of 1000 networks were obtained by independent simulations with randomly chosen outlet positions and random initial surfaces. This model leads to a less proper power law distribution that looks similar to that of Takayasu and Inaoka [3]. Depending on the range of fitting, power laws with various exponents between about 0.4 and 0.5 can be fitted,

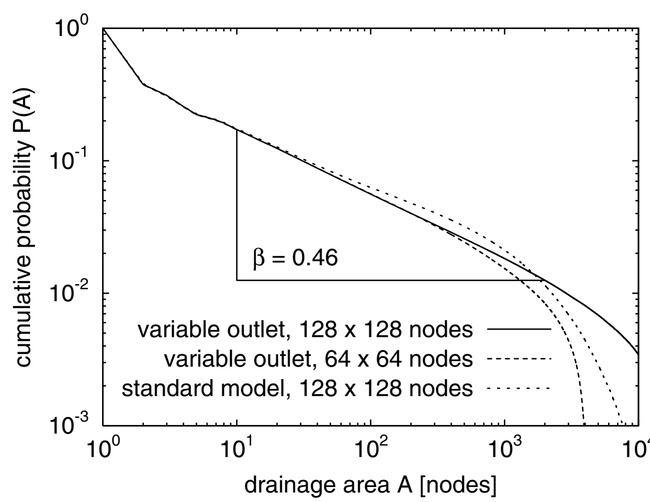

FIG. 2. Size distribution of the drainage areas.

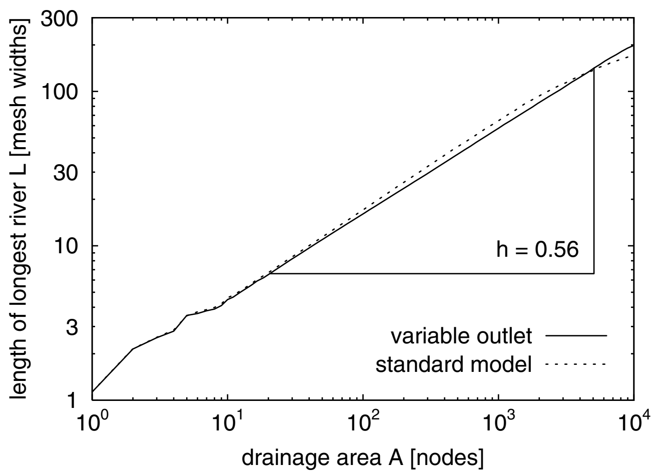

FIG. 3. Hack's law.

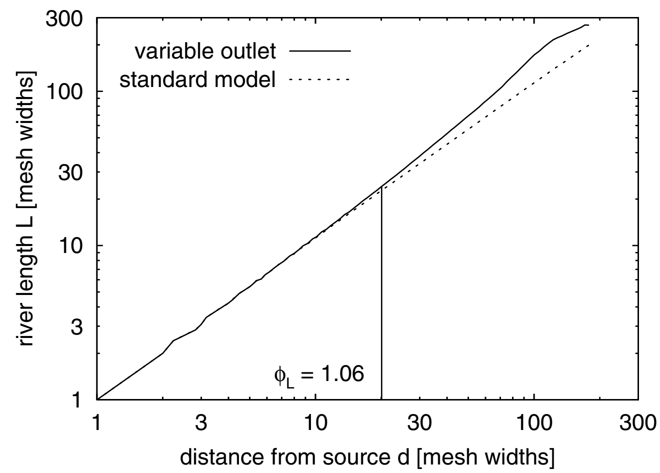

FIG. 4. Length scaling of the rivers.

so that their result $\beta = 0.43$ which is just in the middle of the observed range should not be overinterpreted.

The model reproduces Hack's law: $L \sim A^h$, where L is the length of the longest river within the drainage area A, too. Figure 3 shows the average length of the longest rivers, averaged over small bins. The best-fit power law leads to $h = 0.56$ in agreement with the observed range $h \in [0.52, 0.6]$ [2]. Again, the power law relation is cleaner than in the standard model.

Figure 4 shows the length scaling of rivers, i.e., the relation between the upstream length L of the convoluted rivers and the distance d between the considered point and the river's source. Observations suggest $L \sim d^{\phi_L}$, where $\phi_L \in [1.02, 1.12]$ [2]. The model yields $\phi_L = 1.06$; the convolution of the large rivers is slightly exaggerated compared to the power law.

So far we have obtained a model that predicts the fractal properties of drainage networks and leads to permanently changing patterns, but the fractal properties are still static. In contrast, recognizing SOC behavior requires a fractal distribution of event sizes.

While the surface evolves continuously, the evolution of the drainage network, especially of drainage areas, is discontinuous. Whenever a node changes its flow direction, its whole drainage area may switch from one outlet to the other. Thus, the drainage divide between old and new outlet migrates in discrete steps during the transition from one steady state to another. From this point of view, we can interpret every change in network topology as an event. We define the size of an event as the resulting fluctuation in drainage area, i.e., the size A_c of the area which switches from the old outlet to the new one as a result of a change in drainage direction at one node.

While A_c can be directly obtained by tracing the discharge of the new outlet, distinguishing individual events requires a more careful numerical treatment than computing equilibrated networks, especially an adaptive variation of the time step length. As soon as more than one node changes its flow direction within one time step, two or more events are mixed up, except for the situation that some surface heights coincide. In case of such a mixing, the time step must be rejected and be replaced by some smaller steps until the events are properly separated.

Figure 5 shows the size distribution of these fluctuations in drainage area, monitored over all 5000 transitions. Except for finite size effects, they obey power law statistics: $P(A_c) \sim A_c^{-b}$. The exponent $b = 0.70$ differs significantly from that of the static drainage area distribution ($\beta = 0.46$). Thus, this distribution of event sizes is not just a reflection of the static distribution, but reveals properties of the dynamic organization process. However, a verification of this result in field might be impossible.

Thus, our model of drainage network evolution meets the criteria of SOC. However, we must be aware that the framework of SOC still lacks a clear definition how events and their sizes have to be defined. We could also choose another quantity to be the event size, and it is not clear at all whether we obtained a power law distribution then,

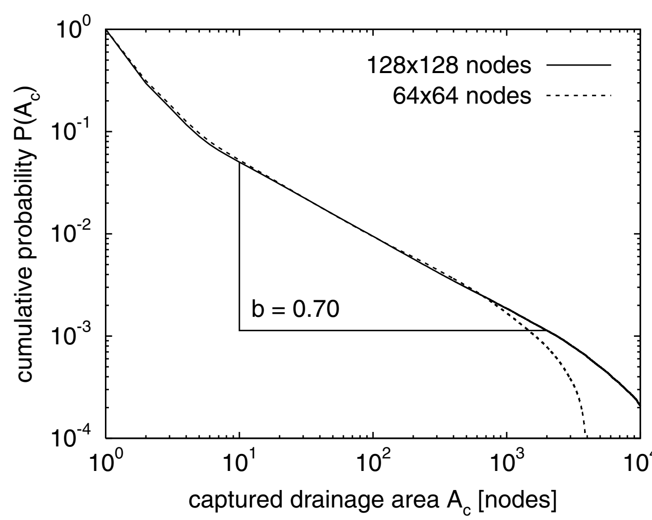

FIG. 5. Size distribution of the fluctuations in drainage areas.

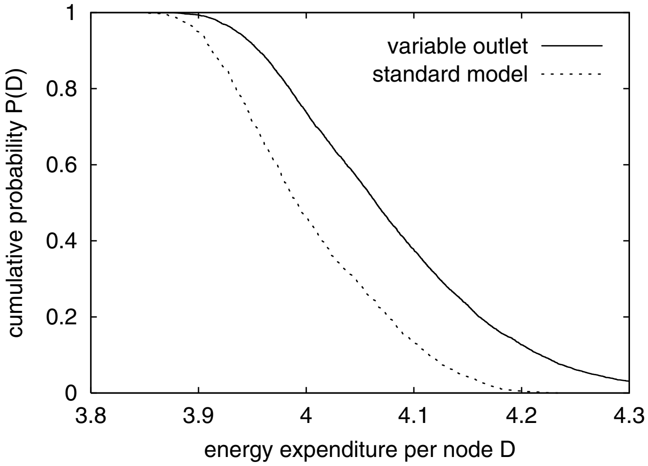

FIG. 6. Distribution of the energy expenditure per node.

too. Moreover, we could switch to a longer time scale and consider the transition from one steady state to another as an event, and we cannot tell whether any property of these events exhibits power law statistics.

Except for the length scaling of the rivers, our model leads to cleaner fractal properties than the standard model does, comparably to those of optimal channel networks. This might be a direct consequence of the permanent reorganization; our networks may simply have more chances to come close to the state of minimum energy expenditure than those emerging from the standard model. However, the following analysis shows that this is not the reason: The normalized average energy dissipation per node is the difference between the mean surface height (over the whole lattice) and the outlet's height. Figure 6 shows the cumulative distribution of this property, obtained from all 5000 networks, compared to that of the standard model. Our model even leads to a higher energy dissipation than the latter does, so our SOC networks are far away from minimum energy expenditure.

From theory of SOC, this behavior is not really surprising because SOC systems tend to approach a strange attractor in phase space rather than a distinct, optimized state. We suspect that there may be a relation between the location of the strange attractor and the state of minimum energy expenditure, but this is not clear yet.

Conclusions and open questions.—Starting from local erosion rules and time-dependent boundary conditions we have presented a model that explains the fractal properties of drainage networks as a result of a SOC process. The SOC behavior proposesthe temporal variations in drainage area as a novel, dynamic scaling property of river networks.

Our self-organized critical drainage networks do not minimize energy expenditure, so our model is not just a physical realization of the principle of minimum energy expenditure. Nevertheless, there may be a relation between the strange attractor in self-organized critical network evolution and the state of minimum energy expenditure which may finally help us to understand the relevance of optimization in drainage network evolution.

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