Nonadditivity of Bipartite Distillable Entanglement Follows from a Conjecture on Bound Entangled Werner States

Peter W. Shor,[1,*] John A. Smolin,[2,†] and Barbara M. Terhal[2,‡]
[1]AT&T Research, Florham Park, New Jersey 07932
[2]IBM Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598
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Assuming the validity of a conjecture given by DiVincenzo *et al.* [Phys. Rev. A **61**, 062312 (2000)] and by Dür *et al.* [Phys. Rev. A **61**, 062313 (2000)], we show that the distillable entanglement for two bipartite states, each of which individually has zero distillable entanglement, can be nonzero. We show that this also implies that the distillable entanglement is not a convex function. Our example consists of the tensor product of a bound entangled state based on an unextendible product basis with an entangled Werner state which lies in the class of conjectured undistillable states.

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One of the central goals of the theory of bipartite quantum entanglement is to develop measures of quantum entanglement. For pure states, this problem is largely solved. One can formulate a set of basic requirements [1] which give rise to a unique measure [2,3] which is the von Neumann entropy $S(\rho) = -\mathrm{Tr}\rho\log\rho$ of the reduced density matrix $\rho = \mathrm{Tr}_A|\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle$. For mixed states, all measures that obey the desirable requirements have been shown to lie between the regularized entanglement of formation $E_\infty(\rho) = \lim_{n\to\infty} E_f(\rho^{\otimes n})/n$, where $E_f(\rho)$ is the entanglement of formation of ρ [4], and the distillable entanglement $D(\rho)$ (see Refs. [4,5] for proper definitions of D). The special role of E_∞ and D among the possible entanglement measures for mixed states is emphasized by the fact that they have a direct physical interpretation; they measure the entanglement costs of making the state ρ asymptotically from pure states [4,6] and the amount of pure entanglement that can be extracted from ρ asymptotically, respectively. Even though these measures are of central importance in the theory of bipartite entanglement, various open questions exist about their basic properties.

There exists one class of bipartite density matrices for which it is known that even though a state ρ in this class is entangled, the distillable entanglement $D(\rho) = 0$. This class of states is characterized by the fact that the states do not violate the Peres-Horodecki criterion, i.e., $(\mathbf{1} \otimes T)(\rho) \geq 0$, where T is matrix transposition in a chosen basis. It was shown in Ref. [7] that this implies that $D(\rho) = 0$. Let us call these states PPT ("positive partial transpose") bound entangled states. Researchers have considered whether this kind of bound entangled state can play a role in quantum information processing; for example, it can be proved that PPT bound entangled states are a useless resource in protocols of quantum teleportation [8] and also superdense coding [9]. On the other hand, it has been found that bound entanglement can be used to quasidistill a single free entangled state [10], something which is not feasible without this additional resource. In this Letter we present an even stronger effect that bound entangled states can have; states which are conjectured to be undistillable become distillable by adding PPT bound entanglement. Let us refer to these conjectured bound entangled states as NPT ("negative partial transpose") bound entangled states. This family of states was considered in Refs. [11] and [12]. The states do violate the Peres-Horodecki criterion; however, they seem to lose this property when trying to squeeze the entanglement (distill) into a smaller set of states. Let us state the conjecture which was made in Refs. [11] and [12]:

Conjecture 1 [11,12].—Given is the class of Werner states [13] in $\mathcal{H}_3 \otimes \mathcal{H}_3$:

$$\rho_W(\lambda) = \frac{1}{8\lambda - 1}\left(\lambda\mathbf{1} - \frac{\lambda + 1}{3} H\right). \tag{1}$$

Here H *is the swap operator, i.e.,* $H|i,j\rangle = |j,i\rangle$ *for all states* $|i,j\rangle$ *where* $i,j = 1,\ldots,3$. *The state* $\lim_{\lambda\to\infty}\rho_W(\lambda)$ *is separable and for any finite* $\lambda \geq 0$ $\rho_W(\lambda)$ *is entangled and violates the Peres-Horodecki criterion. It is conjectured that for all* $\lambda \geq 2$ *the state* $\rho_W(\lambda)$ *is undistillable, i.e.,* $D[\rho_W(\lambda)] = 0$.

Before reviewing the evidence for this conjecture, let us recall the condition for distillability:

Theorem 1 [7,11,12].—The density matrix ρ *is distillable, i.e.,* $D(\rho) > 0$, *if and only if there exists an* $n > 0$ *such that*

$$\mathrm{Tr}[|\psi_2\rangle\langle\psi_2|(\mathbf{1} \otimes T)(\rho^{\otimes n})] < 0, \tag{2}$$

where $|\psi_2\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ *is a state with Schmidt rank 2 and* T *is matrix transposition in any basis.*

The evidence in support of the conjecture is the following. If we set $n = 1$ in Theorem 1, one can prove that Eq. (2) is non-negative for all states $\rho_W(\lambda)$ with $\lambda \geq 2$. Furthermore, for $n = 2$ and $n = 3$ numerical evidence for the non-negativity for Eq. (2) has been found for these states. Also it has been proved that for every finite n in Theorem 1, there exists a finite λ for which Eq. (2) is not satisfied. The evidence, even though it is convincing, is not conclusive.

In this Letter we consider the distillability properties of a pair of states, one of which has PPT bound entanglement and one which has (conjectured) NPT bound entanglement. Surprisingly we find that the distillable entanglement of the pair can be nonzero. Thus

$$D(\rho_1 \otimes \rho_2) > 0, \quad D(\rho_1) = 0, \quad D(\rho_2) \overset{\text{conjectured}}{=} 0, \tag{3}$$

which is an extreme example of *nonadditivity* of the distillable entanglement known as "superactivation" [14] assuming that the conjecture holds. Examples of superactivation of bound entanglement have previously been found in a multipartite system [14]. The strict superaddivity that we find here seems even more surprising since we expect that fewer incomparable resources and states exist in the bipartite case.

This nonadditivity has an added consequence, namely, the entanglement measure D will not be convex if Conjecture 1 holds. Let us take the states ρ_1 and ρ_2 for which Eq. (3) holds and mix them in this way:

$$\rho = \frac{1}{2}\rho_1 \otimes (|1\rangle\langle 1|)_A + \frac{1}{2}\rho_2 \otimes (|2\rangle\langle 2|)_A. \tag{4}$$

Convexity of D would imply that

$$D(\rho) \leq \frac{1}{2}D[\rho_1 \otimes (|1\rangle\langle 1|)_A] + \frac{1}{2}D[\rho_2 \otimes (|2\rangle\langle 2|)_A] \overset{\text{conjectured}}{=} 0. \tag{5}$$

However, we can show that $D(\rho) > 0$. To distill the mixture, Alice first measures the label $|1\rangle$ and $|2\rangle$ on many copies of ρ. This will give Alice and Bob a supply of both ρ_1 as well as ρ_2 which can be distilled since $D(\rho_1 \otimes \rho_2) > 0$. We must conclude that demanding convexity of an entanglement measure, as was done in Ref. [1], is too constraining [15].

Another consequence of the result is a nonzero lower bound on the entanglement of formation of ρ_2. From Proposition 3 in Ref. [16] we have that the distillable entanglement of ρ_2, assisted by bound entanglement [e.g., state ρ_1 for which $D(\rho_1) = 0$] is a lower bound for the regularized entanglement of formation $E_\infty(\rho_1)$, or $E_\infty(\rho_1) \geq D(\rho_1 \otimes \rho_2) > 0$. Note that if Conjecture 1 holds, the same is true for state ρ_1, i.e., $D(\rho_1) = 0$, but $E_\infty(\rho_1) \geq D(\rho_1 \otimes \rho_2) > 0$ which would provide an additional example of irreversible asymptotic entanglement processing [17].

The distillable state $\rho = \rho_1 \otimes \rho_2$ also provides the first nontrivial example of a density matrix which satisfies the reduction criterion [18], i.e., $\mathbf{1}_A \otimes \rho_B - \rho \geq 0$ and $\rho_A \otimes \mathbf{1}_B - \rho \geq 0$, while it is distillable. This follows from the fact that both ρ_1 and ρ_2 satisfy the reduction criterion (otherwise they would be distillable) and the fact that any tensor product of states that by itself satisfies the criterion satisfies the reduction criterion as well [18].

For our PPT bound entangled state we choose a bound entangled state in $\mathcal{H}_3 \otimes \mathcal{H}_3$ based on an unextendible product basis (UPB) [19]. In particular, in Ref. [19] the **Pyramid** UPB was introduced and the corresponding bound entangled state $\rho_{\mathbf{Pyr}}$. The unextendible product basis is given by five vectors

$$|v_i \otimes v_{2i \bmod 5}\rangle, \qquad i = 0, \ldots, 4, \tag{6}$$

where

$$|v_i\rangle = N[\cos(2\pi i/5), \sin(2\pi i/5), h], \tag{7}$$

and $N = 2/\sqrt{5 + \sqrt{5}}$ and $h = \frac{1}{2}\sqrt{1 + \sqrt{5}}$. The bound entangled state $\rho_{\mathbf{Pyr}}$ is equal to

$$\rho_{\mathbf{Pyr}} = \frac{1}{4}\left(\mathbf{1} - \sum_{i=0}^{4}|v_i, v_{2i \bmod 5}\rangle\langle v_i, v_{2i \bmod 5}|\right). \tag{8}$$

Our choice of the NPT bound entangled state is the Werner state $\rho_W(\lambda)$ in $\mathcal{H}_3 \otimes \mathcal{H}_3$. The partial transpose of this state is

$$(\mathbf{1} \otimes T)[\rho_W(\lambda)] = \frac{1}{8\lambda - 1}[\lambda\mathbf{1} - (\lambda + 1)|\Psi\rangle\langle\Psi|], \tag{9}$$

where $|\Psi\rangle = \frac{1}{\sqrt{3}}\sum_i |ii\rangle$.

We show that there exists a vector $|\psi_2\rangle$ which has Schmidt rank 2 with the property

$$\mathrm{Tr}\{|\psi_2\rangle\langle\psi_2|[\mathbf{1} \otimes T][\rho_W(\lambda) \otimes \rho_{\mathbf{Pyr}}]\} < 0, \tag{10}$$

for a certain range in λ. From Theorem 1 it then follows that $\rho_W(\lambda) \otimes \rho_{\mathbf{Pyr}}$ is distillable. The vector $|\psi_2\rangle \in \mathcal{H}_{A_1,B_1} \otimes \mathcal{H}_{A_2,B_2}$ can be parametrized as

$$|\psi_2\rangle = \sum_{i,j}|i,j\rangle \otimes |\psi_{ij}\rangle, \tag{11}$$

where the vectors $|\psi_{ij}\rangle$ are of the form

$$|\psi_{ij}\rangle = |x_i\rangle \otimes |y_j\rangle + |z_i\rangle \otimes |u_j\rangle, \tag{12}$$

due to the fact that $|\psi_2\rangle$ has Schmidt rank 2 over a cut in A_1, A_2 versus B_1 and B_2. Here the vectors $|x_i\rangle$, $|y_i\rangle$, $|z_i\rangle$, $|u_i\rangle$ are unnormalized arbitrary vectors in $\mathcal{H}_3$, to be fixed later. We will not be concerned with the normalization of the vector $|\psi_2\rangle$ since this is irrelevant for the sign in Eq. (10).

It was noted in Ref. [19] that the density matrix $\rho_{\mathbf{Pyr}}$ is invariant under partial transposition $\mathbf{1} \otimes T$. Using this fact and the parametrization of $|\psi_2\rangle$ we can express Eq. (10) [dropping the factor $1/(8\lambda - 1)$] in terms of the vectors $|\psi_{ij}\rangle$:

$$\mathrm{Tr}[\lambda\textstyle\sum_i|\psi_{ii}\rangle\langle\psi_{ii}| - \frac{\lambda+1}{3}\sum_{i,j}|\psi_{ii}\rangle\langle\psi_{jj}|]\rho_{\mathbf{Pyr}} + \lambda\,\mathrm{Tr}\textstyle\sum_{i\neq j}|\psi_{ij}\rangle\langle\psi_{ij}|\rho_{\mathbf{Pyr}}. \tag{13}$$

We make a choice for the vectors $|\psi_{ij}\rangle$ which results in

$$|\psi_2\rangle = 2|0,2\rangle \otimes |v_4, v_3\rangle + \tfrac{1}{2}|2,0\rangle \otimes |v_3, v_1\rangle + 2|1,2\rangle \otimes |v_1, v_2\rangle + \tfrac{1}{2}|2,1\rangle \otimes |v_2, v_4\rangle + |0,0\rangle \otimes |v_4, v_1\rangle - |1,1\rangle \otimes |v_1, v_4\rangle + |2,2\rangle \otimes (|v_3, v_3\rangle - |v_2, v_2\rangle). \quad (14)$$

It can easily be checked that this choice corresponds to setting

$$\begin{cases} -|x_1\rangle = |z_1\rangle = 2|y_0\rangle = 2|u_0\rangle = |v_1\rangle \\ |x_0\rangle = |z_0\rangle = 2|y_1\rangle = -2|u_1\rangle = |v_4\rangle \\ 2|x_2\rangle = |u_2\rangle = |v_3\rangle + |v_2\rangle \\ 2|z_2\rangle = |y_2\rangle = |v_3\rangle - |v_2\rangle. \end{cases} \quad (15)$$

Now we observe the consequences for Eq. (13). Since we have chosen the states $|\psi_{ij}\rangle$ for $i \neq j$ to be equal to either the zero vector or one of the UPB vectors, Eq. (6), we ensure that the last term in Eq. (13) is 0. We then use that the inner products of the remaining vectors with respect to the state $\rho_{\mathbf{Pyr}}$ are given by

$$\begin{aligned} \langle v_1, v_4|\rho_{\mathbf{Pyr}}|v_1, v_4\rangle &= \frac{\sqrt{5}}{2} - 1, \\ \langle v_4, v_1|\rho_{\mathbf{Pyr}}|v_4, v_1\rangle &= \frac{\sqrt{5}}{2} - 1, \\ \langle v_4, v_1|\rho_{\mathbf{Pyr}}|v_1, v_4\rangle &= \frac{-7 + 3\sqrt{5}}{8}, \\ \langle \psi_{22}|\rho_{\mathbf{Pyr}}|\psi_{22}\rangle &= \sqrt{5} - 2 - \frac{3 - \sqrt{5}}{4}, \\ \langle \psi_{22}|\rho_{\mathbf{Pyr}}|v_4, v_1\rangle &= \frac{-2 + \sqrt{5}}{4}, \\ \langle \psi_{22}|\rho_{\mathbf{Pyr}}|v_1, v_4\rangle &= \frac{2 - \sqrt{5}}{4}. \end{aligned} \quad (16)$$

Here $|\psi_{22}\rangle = |v_3, v_3\rangle - |v_2, v_2\rangle$. Hence it follows that Eq. (13) equals

$$\langle \psi_2|[\mathbf{1} \otimes T][\rho_W(\lambda) \otimes \rho_{\mathbf{Pyr}}]|\psi_2\rangle = \frac{1}{12}[\lambda(17\sqrt{5} - 37) + 20 - 10\sqrt{5}]. \quad (17)$$

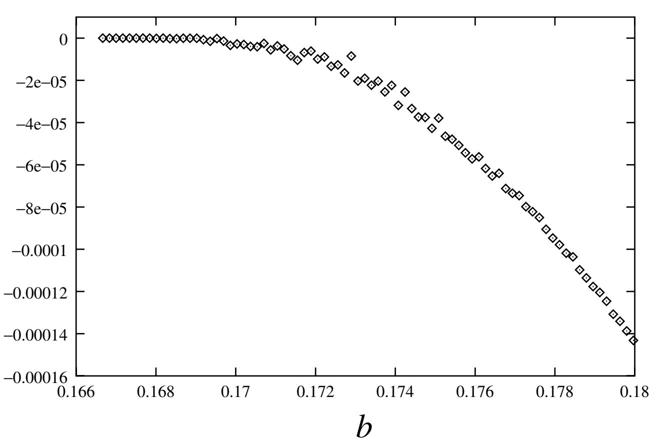

FIG. 1. Numerical results on the value of Eq. (10) versus b. At $b = \frac{1}{6}$ the density matrix $\rho_W(\lambda)$ is separable. When $b > \frac{1}{5}$ $\rho_W(\lambda)$ is distillable.

This expression is negative when

$$\lambda < \frac{10\sqrt{5} - 20}{17\sqrt{5} - 37} \approx 2.3300. \quad (18)$$

Thus this solution provides a proof that in the range $\lambda \in [2, 2.3300)$ the state $\rho_{\mathbf{Pyr}} \otimes \rho_W(\lambda)$ is distillable.

The solution that we have constructed analytically may not be optimal. We have carried out a numerical study, see Fig. 1, evaluating the minimum value of Eq. (10) while varying the parameter b which is related to λ by $\lambda = (b + 1/3)/(8b - 4/3)$, or $b \in (1/6, 1/5]$ when $\lambda \in (\infty, 2]$. As the figure shows, the activation effect is extremely small (all density matrices and states are normalized here, unlike in the analytical procedure above) and seems to vanish before we reach the boundary with the set of separable Werner states (see also [20]). It is possible that by using two or more states $\rho_{\mathbf{Pyr}}$ for the activation we obtain a negative expectation value for smaller values of b.

The activation of $\rho_W(\lambda)$ by $\rho_{\mathbf{Pyr}}$ is not an effect particular to $\rho_{\mathbf{Pyr}}$. The strategy to minimize Eq. (13) can very likely be generalized to other bound entangled states based on unextendible product bases. We can always put the last term to zero, by choosing the states $|\psi_{ij}\rangle$ to be either 0 vectors or UPB vectors. This gives us some additional constraints for the states $|\psi_{ii}\rangle$, but the number of free parameters will still be quite large.

In conclusion, pending the proof of Conjecture 1, we have determined an essential new and surprising property of the distillable entanglement, namely, its capacity to be nonadditive. It is clear that it would be highly desirable to prove the conjecture, but that goal remains elusive for the moment.

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*Electronic address: shor@research.att.com
†Electronic address: smolin@watson.ibm.com
‡Electronic address: terhal@watson.ibm.com

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