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Non-Abelian Statistics of Half-Quantum Vortices in p-Wave Superconductors

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Excitation spectrum of a half-quantum vortex in a p-wave superconductor contains a zero-energy Majorana fermion. This results in a degeneracy of the ground state of the system of several vortices. From the properties of the solutions to Bogoliubov–de Gennes equations in the vortex core we derive the non-Abelian statistics of vortices identical to that for the Moore-Read (Pfaffian) quantum Hall state.

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Certain types of superconductors with triplet pairing allow half-quantum vortices [1,2]. Such vortices appear if the multicomponent order parameter has extra degrees of freedom besides the overall phase, and the vortex involves both a rotation of the phase by π and a rotation of the "direction" of the order parameter by π, so that the order parameter maps to itself on going around the vortex. The magnetic flux through such a vortex is one-half of the superconducting flux quantum Φ_0.

As far as low-lying subgap excitations are concerned, a half-quantum vortex for spinful fermions is equivalent to a single-quantum vortex in a p-wave superconductor of spinless fermions. A remarkable feature of such a vortex is a Majorana fermion level at zero energy inside the vortex core [3]. This energy level has a topological nature [4] and from continuity considerations must be stable to any local perturbations. In terms of energy levels, the Majorana fermions in vortex cores imply a 2^n-fold degeneracy of the ground state of a system with $2n$ isolated vortices. If the vortices adiabatically move around each other, this motion may result in a unitary transformation in the space of ground states (non-Abelian statistics). We shall see that it is indeed the case.

The non-Abelian statistics for half-quantum vortices has been previously derived for the Pfaffian quantum Hall state proposed by Moore and Read [5]. The Pfaffian state is of Laughlin type corresponding to filling fractions with even denominator. The excitations in the Pfaffian state are half-quantum vortices, and their non-Abelian statistics has been obtained in the field-theoretical framework [6–9].

Recently Read and Green suggested that the Pfaffian state belongs to the same topological class as the BCS pairing state and thus the latter must have the same non-Abelian statistics [3]. In our paper we verify this directly in the BCS framework as the property of solutions to Bogoliubov–de Gennes equations. Our derivation provides an alternative (and possibly more transparent) point of view on the non-Abelian statistics of half-quantum vortices as well as an additional confirmation of topological equivalence between Pfaffian and BCS states.

First, we point out the equivalence between a half-quantum vortex for spinful fermions and a single-quantum vortex for spinless (or spin-polarized) fermions. Consider first the spinless case. To respect the anticommutativity of fermions, the superconducting order parameter in a spinless superconductor must have odd parity. We consider a two-dimensional superconductor with a chiral order parameter $\Delta(k) = (k_x \pm ik_y)\Delta$. The ± sign denotes the two possible chiralities of the condensate. In this paper we do not discuss interaction of vortices with domain walls separating regions of opposite chirality. Instead, we assume that the chirality is fixed in the region where the vortex braiding occurs (e.g., an external field can make one chirality energetically favorable). The chirality breaks time-reversal symmetry, and the positive and negative vortices are not equivalent. The two types of vortices have slightly different structure of the quasiparticle eigenfunctions, but their low-energy spectra and the braiding statistics are the same. Our discussion below is applicable to both positive and negative vortices.

The Hamiltonian of the axially symmetric (positive) vortex is

$$H = \int d^2\mathbf{r}\Big[\Psi^\dagger\Big(-\frac{\nabla^2}{2m} - \varepsilon_F\Big)\Psi + \Psi^\dagger[e^{i\theta}\Delta(r) * (\nabla_x + i\nabla_y)]\Psi^\dagger + \text{H.c.}\Big], \tag{1}$$

where $*$ is the symmetrized product $[A * B = (AB + BA)/2]$, r and θ are the polar coordinates.

The same vortex Hamiltonian (1) describes the low-energy excitations in the half-quantum vortex in a chiral p-wave superconductor of spinful fermions (with the order parameter of A phase of ^{3}He). In the spinful case, the order parameter is characterized not only by its phase φ, but also by the direction $\hat{\mathbf{d}}$ of triplet pairing. The wave function of the condensate is

$$\Psi = e^{i\varphi}[d_x(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + id_y(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) + d_z(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)](k_x + ik_y)\,. \tag{2}$$

For a half-quantum vortex to exist, the vector $\hat{\mathbf{d}}$ must be able to rotate (either in a plane or in all three dimensions). The order parameter maps to itself under simultaneous change of sign of the vector $\hat{\mathbf{d}}$ and shift of the phase φ

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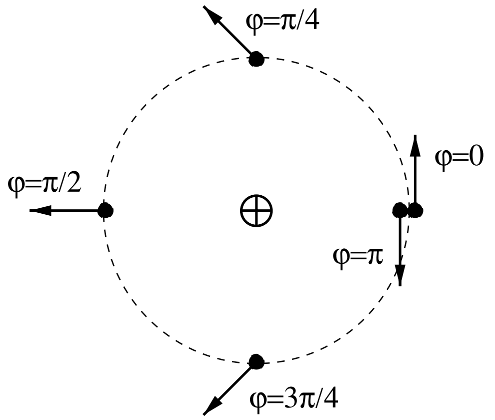

FIG. 1. Half-quantum vortex. Arrows denote the direction of vector $\hat{\mathbf{d}}$.

by π: $(\varphi, \hat{\mathbf{d}}) \mapsto (\varphi + \pi, -\hat{\mathbf{d}})$. The half-quantum vortex combines rotations of the vector $\hat{\mathbf{d}}$ by π and of the phase φ by π on going around the vortex core (Fig. 1). This vortex is topologically stable, i.e., it cannot be removed by a continuous (homotopic) deformation of the order parameter.

Without loss of generality, let vector $\hat{\mathbf{d}}$ rotate in the x-y plane. Then the condensate wave function (2) in the vortex takes the form (in the polar coordinates r and θ):

$$\Psi(r, \theta) = \Delta(r)[e^{i\theta}|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle](k_x + ik_y). \qquad (3)$$

The spin-up and spin-down electrons decouple. The spin-down sector has no vortex and consequently no low-energy states. For our further discussion of adiabatic vortex motion it may be neglected. The spin-up sector is in turn described by the Hamiltonian (1).

The Hamiltonian (1) may be diagonalized by quasiparticle operators $\gamma^\dagger = u\Psi^\dagger + v\Psi$. The Bogoliubov–de Gennes equations $[H, \gamma^\dagger] = E\gamma^\dagger$ for u and v are identical to those for a single-quantum vortex (with the vector $\hat{\mathbf{d}}$ constant in space) and were solved by Kopnin and Salomaa in the context of superfluid ^{3}He vortices [10]. The low-energy spectrum is $E_n = n\omega_0$, where $\omega_0 \sim \Delta^2/\varepsilon_F$ is the level spacing. The quantum number n takes integer values (which distinguishes p-wave vortex states from Caroli–de Gennes–Matricon states in s-wave vortices [11]) and has the meaning of the angular momentum of the quasiparticle.

The half-quantum vortex (1) differs from the conventional single-quantum vortex in that the coefficients u and v correspond not to fermions of opposite spin, but to the creation and annihilation of the same fermion. As a consequence, the solutions to Bogoliubov–de Gennes equations in the half-quantum vortex obey the additional relation between positive- and negative-energy eigenstates $\gamma^\dagger(E) = \gamma(-E)$. In other words, the solutions with positive and negative energies are the creation and annihilation operators for the same fermionic level. Therefore the number of degrees of freedom in a half-quantum vortex is *one-half* of that in the conventional single-quantum vortex. The zero-energy level becomes a self-conjugate (Majorana) fermion:

$$\gamma^\dagger(E = 0) = \gamma(E = 0). \qquad (4)$$

The self-conjugacy condition is specific for the half-quantum vortex in a triplet superconductor and distinguishes it from other systems with topological zero-energy levels such as solitons in polyacetylene [12]. In the half-quantum vortex, the spin degree of freedom is excluded, and the creation and annihilation operators are mixed by superconductivity, making the self-conjugacy relation possible. Formulating general conditions for the occurrence of a Majorana fermionic level in an arbitrary quantum-mechanical system remains an interesting open problem, as well as designing an experimental realization of an isolated Majorana fermion.

It is worth mentioning that the Majorana fermion in the half-quantum vortex (and the non-Abelian statistics derived below) is stable with respect to any local perturbation including external potential, electromagnetic vector potential, local deformations of the order parameter, spin-orbit interaction, and Zeeman splitting (in a single-quantum vortex, only the first three of those perturbations preserve the zero-energy level [13]). We can easily prove it with continuity considerations. Indeed, suppose that we gradually increase perturbation to the vortex Hamiltonian (which includes both the spin-up and spin-down sectors). The levels will shift and mix, but they must do it continuously, and therefore the number of levels is preserved. Since it is half-integer without perturbation, it must remain half-integer for the perturbed Hamiltonian, i.e., the Majorana fermion survives the perturbation. This argument is valid as long as the perturbation is sufficiently small so that the low-lying states remain localized in the vortex.

Before we turn to discussing the non-Abelian statistics of vortices, let us see how the Majorana fermion $\gamma(E = 0)$ transforms under $U(1)$ gauge transformations. If the overall phase of the superconducting gap shifts by ϕ, it is equivalent to rotating electronic creation and annihilation operators by $\phi/2$: $\Psi_\alpha \mapsto e^{i\phi/2}\Psi_\alpha$, $\Psi_\alpha^\dagger \mapsto e^{-i\phi/2}\Psi_\alpha^\dagger$. The solution (u, v) transforms accordingly: $(u, v) \mapsto (ue^{i\phi/2}, ve^{-i\phi/2})$. The important consequence of this transformation rule is that under change of the phase of the order parameter by 2π the Majorana fermion in the vortex changes sign: $\gamma \mapsto -\gamma$. This is an obvious consequence of the fact that the quasiparticle is a linear combination of fermionic creation and annihilation operators carrying charge ± 1.

Now consider a system of $2n$ vortices, far from each other (at distances much larger than $\xi_0 \sim v_F/\Delta$). To each vortex there corresponds one Majorana fermion (further we shall denote them by γ_i, $i = 1, \dots, 2n$) commuting with the Hamiltonian. They can be combined into n complex fermionic operators and therefore give rise to the degeneracy of the ground state equal to 2^n (each fermionic level may be either filled or empty). If the vortices move adiabatically slowly so that we can neglect transitions between subgap levels, the only possible effect of such vortex motion is a unitary evolution in the space of ground states.

Let us fix the initial positions of vortices. Consider now a permutation (braiding) of vortices which returns vortices to their original positions (possibly in a different order). Such braid operations form a braid group B_{2n} (multiplication in this group corresponds to the sequential application of the two braid operations) [14]. This group is generated by elementary interchanges T_i of neighboring particles ($i = 1, \dots, 2n - 1$) modulo the relations (see Fig. 2):

$$\begin{aligned} T_i T_j &= T_j T_i, \qquad |i - j| > 1, \\ T_i T_j T_i &= T_j T_i T_j, \qquad |i - j| = 1. \end{aligned} \tag{5}$$

The braiding statistics is defined by the unitary operators in the space of ground states representing the braid operations from B_{2n}. Here an important reservation has to be made. When a vortex moves along a closed loop, the multiparticle state acquires a phase proportional to the area inside the loop (every electron inside the loop effectively moves around the vortex). We shall disregard this effect and, as a consequence, lose information about the overall phase of the wave function. In other words, we shall speak about only a *projective* representation of the braid group B_{2n}. However, since the representation is multidimensional, the resulting projective representation is still nontrivial and transforms different states into each other—which implies the non-Abelian statistics of vortices.

Since the Majorana fermions γ_i change sign under a shift of the superconducting phase by 2π, we introduce *cuts* connecting vortices to the left boundary of the system (Fig. 3). We take the superconducting phase single valued away from the cuts and jumping by 2π across the cuts. From examining Fig. 3 one easily obtains that the transformation exchanging the two vortices i and $i + 1$ (with no vortices between them) changes the phase of the order parameter at one of the vortices by 2π, which results in the following transformation rule:

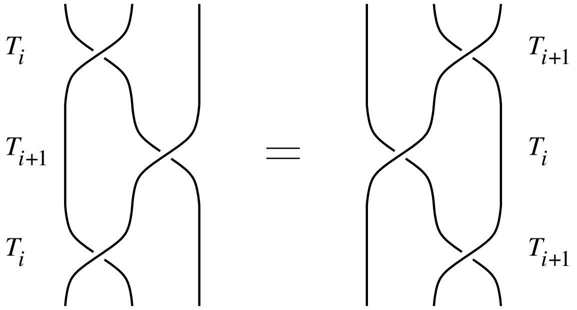

FIG. 2. Defining relation for the braid group: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

$$T_i : \begin{cases} \gamma_i \mapsto \gamma_{i+1}, \\ \gamma_{i+1} \mapsto -\gamma_i, \\ \gamma_j \mapsto \gamma_j \qquad \text{for } j \neq i \text{ and } j \neq i + 1. \end{cases} \tag{6}$$

This defines the action of T_i on Majorana fermions. One easily checks that this action obeys the commutation relations (5).

Now the action of operators T_i may be extended from *operators* to the Hilbert space. Since the whole Hilbert space can be constructed from the vacuum state by fermionic creation operators, and the mapping of the vacuum state by T_i may be determined uniquely up to a phase factor, the action (6) of B_{2n} on operators uniquely defines a projective representation of B_{2n} in the space of ground states.

The explicit formulas for this representation may be written in terms of fermionic operators. Namely, we need to construct operators $\tau(T_i)$ obeying $\tau(T_i)\gamma_j[\tau(T_i)]^{-1} = T_i(\gamma_j)$, where $T_i(\gamma_j)$ is defined by (6). If we normalize the Majorana fermions by $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, then the expression for $\tau(T_i)$ is

$$\tau(T_i) = \exp\left(\frac{\pi}{4}\gamma_{i+1}\gamma_i\right) = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i) \tag{7}$$

(up to a phase factor).

This formula presents the main result of our calculation. On inspection, this representation coincides with that described by Nayak and Wilczek for the statistics of the Pfaffian state [6] (our Majorana fermions correspond to the operators γ_i in section 9 of their paper).

The two simplest examples of the representation (7) are the cases of two and four vortices. These examples were previously discussed to some extent in the Pfaffian framework in Refs. [6,7], and we review them here for illustration purposes.

In the case of two vortices, the two Majorana fermions may be combined into a single complex fermion as $\Psi = (\gamma_1 + i\gamma_2)/2$, $\Psi^\dagger = (\gamma_1 - i\gamma_2)/2$. The ground state is doubly degenerate, and the only generator of the braid group T is represented by

$$\begin{aligned} \tau(T) &= \exp\left(\frac{\pi}{4}\gamma_2\gamma_1\right) = \exp\left[i\frac{\pi}{4}(2\Psi^\dagger\Psi - 1)\right] \\ &= \exp\left(i\frac{\pi}{4}\sigma_z\right), \end{aligned} \tag{8}$$

where σ_z is a Pauli matrix in the basis $(|0\rangle, \Psi^\dagger|0\rangle)$.

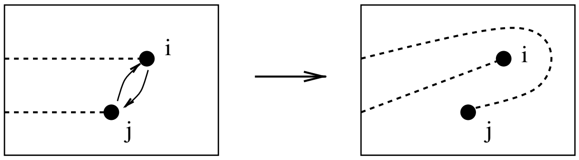

FIG. 3. Elementary braid interchange of two vortices.

In the case of four vortices, the four Majorana fermions combine into two complex fermions Ψ_1 and Ψ_2 by $\Psi_1 = (\gamma_1 + i\gamma_2)/2$, $\Psi_2 = (\gamma_3 + i\gamma_4)/2$ (and similarly for $\Psi_1^\dagger$ and $\Psi_2^\dagger$). The ground state has degeneracy four, and the three generators T_1, T_2, and T_3 of the braid group are represented by

$$\begin{aligned}
\tau(T_1) &= \exp\left(i\,\frac{\pi}{4}\,\sigma_z^{(1)}\right) = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{i\pi/4} & & \\ & & e^{-i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}, \\
\tau(T_3) &= \exp\left(i\,\frac{\pi}{4}\,\sigma_z^{(2)}\right) = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}, \\
\tau(T_2) &= \exp\left(\frac{\pi}{4}\,\gamma_3\gamma_2\right) = \frac{1}{\sqrt{2}}(1 + \gamma_3\gamma_2) = \frac{1}{\sqrt{2}}[1 + i(\Psi_2^\dagger + \Psi_2)(\Psi_1^\dagger - \Psi_1)] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{9}$$

where the matrices are written in the basis ($|0\rangle$, $\Psi_1^\dagger|0\rangle$, $\Psi_2^\dagger|0\rangle$, $\Psi_1^\dagger\Psi_2^\dagger|0\rangle$).

There are two important properties of the representation (7) [6,7]. The first one is that $\tau(T)$ are even in fermionic operators and therefore preserve the parity of the number of fermions (physically, this simply means that the superconducting Hamiltonian creates and destroys electrons only in pairs). Therefore the representation may be restricted to odd or even sector of the space of ground states, each of them containing 2^{n-1} states (this degeneracy was also found for the Pfaffian state in Refs. [6–8]). Still, in each of these subspaces the representation operators are nontrivial and noncommuting.

The second property of the representation (7) is that T_i^4 is represented by a scalar matrix (projectively equivalent to the unity matrix, since we disregard the overall phase). That is, an elementary interchange of two vortices repeated four times produces an identity operator (up to an overall phase).

Quite remarkably, our derivation of the non-Abelian statistics relies only on the two facts: first, the flux quantization (half quantum for spin-1/2 electrons or, equivalently, single quantum for spinless fermions) and, second, that the Majorana fermions carry odd charge with respect to the vortex gauge field, i.e., they transform as $\gamma_i \mapsto -\gamma_i$ when the phase of the order parameter changes by 2π. But these are quantization properties that depend only on the presence of the Majorana fermion in the vortex spectrum, but not on the exact form of the Hamiltonian. Therefore, if we introduce disorder or other local perturbation in the BCS Hamiltonian (such as electromagnetic vector potential, spin-orbit scattering, or local deformation of the order parameter), then not only the Majorana fermions survive, but also the braiding statistics (7) remains unchanged (provided the Majorana fermions stay localized in vortices). Thus we may speak of the topological stability of the non-Abelian statistics (7).

Finally, we mention that the operators $\tau(T_i)$ have also been discussed in the context of quantum computation as part of a universal set of operators [15]. Also, non-Abelian anyons provide a topologically stable realization of unitary operators for quantum computing [16]. Thus, should p-wave superconductors with sufficiently large T_c (or, equivalently, large ω_0) be discovered, they may provide a promising hardware solution for quantum computation.

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