Longitudinal and Transverse Waves in Yukawa Crystals

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A unified theoretical treatment is given of longitudinal (or compressional) and transverse modes in Yukawa crystals, including the effects of damping. Dispersion relations are obtained for hexagonal lattices in two dimensions and bcc and fcc lattices in three dimensions. Theoretical predictions are compared with two recent experiments.

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In a Yukawa system, charged microparticles interact with each other through a Yukawa or a screened Coulomb potential. For a point charge Q located at the equilibrium position $\mathbf{R} = \hat{\mathbf{x}}X + \hat{\mathbf{y}}Y + \hat{\mathbf{z}}Z$, this potential is given by $\phi(r) = (Q/r)e^{-r/\lambda_D}$, where $r = |\mathbf{r} - \mathbf{R}| = [(x - X)^2 + (y - Y)^2 + (z - Z)^2]^{1/2}$. Various physical systems, including colloidal crystals and strongly coupled dusty plasmas, can be modeled by the Yukawa potential.

In a dusty plasma, highly charged microparticles are suspended in a gaseous electrical discharge. Such a system is said to be strongly coupled when the coupling constant $\Gamma[\equiv Q^2/(aT)]$ for microparticles is equal to or greater than unity. (Here, T is the temperature of each microparticle, and a is the interparticle spacing.) When $\Gamma \gg 1$, the particles have been shown to crystallize [1–4], that is, organize themselves into an ordered spatial structure that we call a Yukawa lattice.

Plasma crystals allow direct optical imaging of particle motion. Compared with colloidal suspensions, the particles are weakly damped. Particle motion can be excited by laser manipulation. This makes it possible to excite and test the dispersion relations of certain types of longitudinal [5–10] and transverse waves [11–15] predicted by theory. (These waves have antecedents in the theory of one-component plasmas (OCPs) [16–20].) The experimentally measured waves in dusty plasma crystals include the acoustic [21] and lattice waves [22], both of which are longitudinal waves, and, most recently, a transverse wave [23].

In laboratory experiments involving waves in two-dimensional Yukawa lattices (or monolayers), the microparticles form a hexagonal lattice. The waves experience a frictional drag due to the background neutral gas as well as ions. This drag has a significant effect on the dispersive properties of the waves. In order to compare experimental data with theory, it is necessary to develop theoretical models in which the structure of the crystal as well as damping are included as essential elements.

In this Letter, we present a unified analysis of longitudinal and transverse waves in a Yukawa crystal, including the effects of damping. Our work complements and extends a recent theoretical study and molecular dynamics simulation of longitudinal and transverse waves in a Yukawa liquid [24,25]. Our method is based on the harmonic approximation, which is standard in the theory of crystal lattices and has been used in earlier studies of two- and three-dimensional Coulomb crystals [17,26]. We derive dispersion relations for the different waves from a master dispersion relation by considering different regimes of the wave number k and the so-called screening parameter $\kappa \equiv a/\lambda_D$, which is the ratio of the interparticle spacing and the Debye length λ_D. [Here, $\lambda_D = (\lambda_{De}^{-2} + \lambda_{Di}^{-2})^{-1/2}$, where λ_{De} and λ_{Di} are the electron and ion Debye lengths, respectively.] We compare quantitatively the predictions of the theory with available experimental data on two-dimensional monolayers produced in the laboratory. It has been demonstrated experimentally that, in a two-dimensional system, the Yukawa potential is a good approximation for the interaction between microparticles [27]. We assume that this potential also holds in a three-dimensional crystal under microgravity conditions. Based on the analogy with OCPs [28], it is generally believed that in three dimensions the preferred crystal structure is a bcc or fcc lattice, depending on the value of κ. Indeed, there is experimental evidence in three-dimensional laboratory dusty plasmas of the appearance of bcc and fcc structures, mixed with hexagonal structures [1,29]. In anticipation of future microgravity experiments [30] where pure bcc and fcc lattices may be realized, we make predictions for two-dimensional damped waves in three-dimensional bcc and fcc lattices.

For horizontal linear modes, we write $\tilde{\mathbf{r}} = \hat{\mathbf{x}}\xi + \hat{\mathbf{y}}\eta$ as the two-dimensional displacement. The linear displacement obeys the equation of motion,

$$\frac{d^2\tilde{\mathbf{r}}(X,Y,Z,t)}{dt^2} + \nu \frac{d\tilde{\mathbf{r}}(X,Y,Z,t)}{dt} = \frac{Q}{M_d}\mathbf{E}[\mathbf{r}(X,Y,Z),t], \tag{1}$$

where ν represents the frictional drag coefficient, M_d is the mass of a point particle, and the electric field is given by $\mathbf{E} = -\nabla\phi$. We assume that the zx plane is a plane of reflection symmetry of the crystal. Assuming that the components of the linear displacement are independent of Z, we write $(\xi, \eta) = (\xi_0, \eta_0)\exp[i(k_x X + k_y Y) - i\omega t]$. It is then straightforward to show that there are two vibrational normal modes given by the dispersion relation,

$$\omega_\pm(\omega_\pm + i\nu) \equiv \Omega_\pm^2(\mathbf{k},\kappa) = \tfrac{1}{2}[F(\mathbf{k},\kappa) + \overline{F}(\mathbf{k},\kappa)] \pm \{[F(\mathbf{k},\kappa) - \overline{F}(\mathbf{k},\kappa)]^2 - 4G^2(\mathbf{k},\kappa)\}^{1/2}. \tag{2}$$

In Eq. (2), we have redefined the various physical quantities to make them dimensionless. We have rescaled $(\omega,\nu)/\omega_0 \to (\omega,\nu)$, where the frequency $\omega_0 \equiv Q(M_d a^3)^{-1/2}$ is proportional to the dust plasma frequency, $(X,Y,Z,R)/a \to (X,Y,Z,R)$, and $ka \to k$. The functions F, $\overline{F}$, and G represent infinite sums over lattice sites and are given by

$$F(\mathbf{k},\kappa) \equiv 4\left[\sum_{X>0,Z} F(X,0,Z)\sin^2\left(\frac{k_xX}{2}\right) + \sum_{Y>0,Z} F(0,Y,Z)\sin^2\left(\frac{k_yY}{2}\right) + \sum_{X,Y>0,Z} F(X,Y,Z)\,(1 - \cos k_xX\cos k_yY)\right], \tag{3a}$$

$$G(\mathbf{k},\kappa) = 4\sum_{X,Y>0,Z} G(X,Y,Z)\sin k_xX\sin k_yY\,, \tag{3b}$$

$$\overline{F}(\mathbf{k},\kappa) \equiv 4\left[\sum_{X>0,Z} F(0,X,Z)\sin^2\left(\frac{k_xX}{2}\right) + \sum_{Y>0,Z} F(Y,0,Z)\sin^2\left(\frac{k_yY}{2}\right) + \sum_{X,Y>0,Z} F(Y,X,Z)\,(1 - \cos k_xX\cos k_yY)\right], \tag{3c}$$

with the spring constant matrices,

$$F(X,Y,Z) = R^{-5}e^{-\kappa R}[X^2(3 + 3\kappa R + \kappa^2R^2) - R^2(1 + \kappa R)], \tag{4a}$$

$$G(X,Y,Z) = (XY/R^5)e^{-\kappa R}(3 + 3\kappa R + \kappa^2R^2). \tag{4b}$$

In Eqs. (3a)–(3c), the summations over X, Y, and Z are carried out over their entire range except when specified otherwise. In particular, the specification $X > 0$ ($Y > 0$) implies that the summation is carried only over positive values of X (Y).

Equations (2)–(4) give the general dispersion relations in three-dimensional Yukawa crystals. If we set $Z = 0$ and omit the sums over Z, we obtain the two-dimensional dispersion relation for a monolayer. Special cases of the two-dimensional dispersion relation have been discussed earlier in [11] and [22].

Acoustic limit.—We choose $\mathbf{k}$ to be parallel to the x axis. The longitudinal (or compressional) and transverse "acoustic" speeds can then be calculated in the long-wavelength limit from the relation $C_{l,t}^2(\kappa) = [\Omega_{+,-}^2(k,\kappa)/k^2]_{k\to0}$. In the $\kappa \ll 1$ regime, the longitudinal acoustic speed can be expanded as a Frobenius series, $C_l^2 = c_{-2}\kappa^{-2} + c_{-1}\kappa^{-1} + c_{\ln}\ln\kappa + c_0 + c_1\kappa + c_2\kappa^2 + \ldots$. In a three-dimensional lattice, it is straightforward to show analytically from the dispersion relation (2) that $C_l^2 \approx c_{-2}\kappa^{-2}$, where $c_{-2} = \kappa^2C_l^2|_{\kappa\to0} = 8\pi$ (16π) for a bcc (fcc) lattice. However, if we scale ω and ν by the standard plasma frequency $\omega_{pd} = (4\pi n_dQ^2/M_d)^{1/2}$ instead of the frequency $\omega_0 = Q(M_da^3)^{-1/2}$ and note that $n_d = 2/a^3$ ($4/a^3$) for a bcc (fcc) lattice, we obtain the so-called dust-acoustic-wave dispersion relation $\omega_l(\omega_l + i\nu) = C_l^2k^2/\kappa^2$, in agreement with the dispersion relation obtained from other three-dimensional strong-coupling models in the strongly coupled regime ($\Gamma \gg 1$) [7–9,24,25]. This dispersion relation also follows in the limit $k \to 0$, $\kappa \ll 1$ of the exact unified dispersion relation derived in [10] for an infinite sequence of parallel sheets in three-dimensional space interacting via an electrostatic force that depends only on the coordinate perpendicular to the sheets. This dispersion relation has been confirmed in a number of experiments in the moderately strong [31] as well as strong-coupling [21] regimes.

For the two-dimensional hexagonal monolayer, we obtain $c_{-2} = 0$, and $c_{-1} = \kappa C_l^2|_{\kappa\to0} = 4\pi/\sqrt{3}$. In this case, it follows that $C_l \approx (4\pi/\sqrt{3}\,\kappa)^{1/2} = 2.69\kappa^{-1/2}$, which is in agreement with the longitudinal acoustic speed calculated numerically in [11].

A one-dimensional longitudinal acoustic speed, valid for a chain of point particles, can be derived from the dispersion relation (2) by setting $Y = Z = 0$ and omitting the sums over Y and Z. The infinite sum over X then can be carried out exactly to obtain

$$C_l^2 = 2\left(\kappa - \ln(e^\kappa - 1) + \frac{\kappa[e^\kappa(\kappa + 2) - 2]}{2(e^\kappa - 1)^2}\right), \tag{5}$$

which reduces to $C_l^2 = -2\ln\kappa$ in the limit $\kappa \ll 1$. We thus note that the κ dependence of the acoustic speed for compressional modes depends sensitively on the geometry of the crystal: It is proportional to κ^{-1} for a three-dimensional lattice, to $\kappa^{-1/2}$ for a two-dimensional lattice, and to $(-\ln\kappa)^{1/2}$ for a one-dimensional chain.

The acoustic velocity of transverse modes in a two-dimensional hexagonal lattice is also isotropic, as shown numerically in [11]. We can demonstrate this as a special case of the general dispersion relation (2) for the transverse modes in the limit $k \to 0$, $\kappa \ll 1$. We obtain $\omega_t(\omega_t + i\nu) = C_t^2k^2$, with $C_t^2 = c_0 + c_1\kappa + c_2\kappa^2 + \ldots \approx 0.26 - 0.02\kappa^2$. In three-dimensional lattices, we

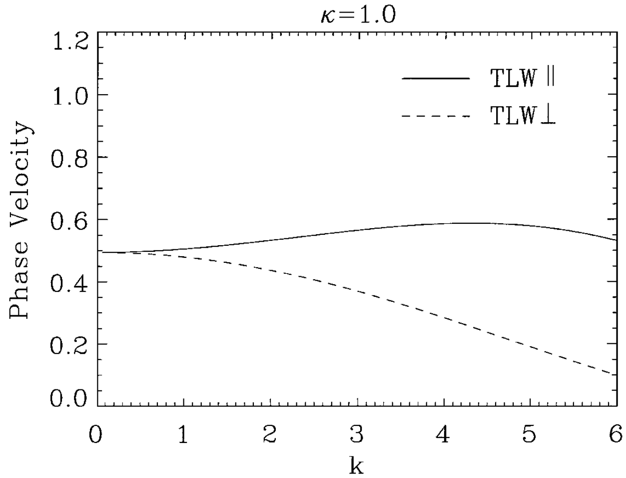

FIG. 1. The phase speed of undamped transverse lattice waves propagating parallel ($\theta = 0$, solid line) and perpendicular ($\theta = \pi/2$, dashed line) to a primitive translation vector.

can calculate the acoustic velocity of transverse modes as a power series in κ, $C_t^2 \approx 0.037 - 0.0013\kappa^2 + 0.000\,018\kappa^4$ which is in approximate agreement with the theoretical calculation for Yukawa liquids [25].

Lattice waves.—We have seen above that the acoustic (or $k \to 0$) limit is isotropic in wave number space. Except in this special limit, the waves are generally anisotropic, and we call them lattice waves. We distinguish two types: the compressional (or longitudinal) lattice wave (CLW) and the transverse lattice wave (TLW). To obtain dispersion relations for lattice waves in hexagonal monolayers, we choose the x axis to be parallel to a primitive translation vector. If θ is the angle between the wave number $\mathbf{k}$ and a primitive translation vector, then $k_x = k\cos\theta$ and $k_y = k\sin\theta$. If $\theta = 0$, the dispersion relation (2) simplifies to produce two branches, where

$$\omega_+(\omega_+ + i\nu) = \omega_l(\omega_l + i\nu) = 2\sum_{X,Y,Z} F(X,Y,Z)\sin^2\left(\frac{kX}{2}\right) \quad (6a)$$

describes the longitudinal modes, and

$$\omega_-(\omega_- + i\nu) = \omega_t(\omega_t + i\nu) = 2\sum_{X,Y,Z} F(Y,X,Z)\sin^2\left(\frac{kX}{2}\right) \quad (6b)$$

describes the transverse modes. The particle motion in these two cases is parallel and perpendicular to $\mathbf{k}$, respectively. Similar dispersion relations can be written for longitudinal and transverse waves when $\theta = \pi/2$, with $\omega_- = \omega_l$ and $\omega_+ = \omega_t$.

In Fig. 1, we plot the phase speeds of the parallel ($\|$) and perpendicular ($\perp$) TLW as a function of k for $\kappa = 1$ when the frictional damping is zero. In the limit $k \to 0$, the phase speeds of the parallel ($\theta = 0$) and perpendicular ($\theta = \pi/2$) TLW tend to the acoustic limit but deviate significantly from this limit for nonzero values of k. The TLW is, in general, strongly anisotropic. In the regime $k \leq 5$, whereas the speed of the parallel TLW increases, that of the perpendicular TLW decreases—a feature that can be tested by changing the direction of manipulation by a laser in an experiment.

Effects of damping and comparison with experiments.—We now consider the effects of frictional damping and demonstrate its important role in experiments. Both the experiments discussed in this Letter use laser pressure on particles to excite waves and measure the real and imaginary parts of the complex wave number $k = k_r + ik_i$. Without damping, one obtains $k_i = 0$ and the waves propagate unattenuated in space. In view of the experimental uncertainties in crucial parameters such as κ and ν, it is important to carry out experimental tests of the real as well as the imaginary part of k predicted by theory.

In Fig. 2, we compare the predictions of the theoretical dispersion relation (6a) with experimental data points (represented by squares) for the approximate parameters given in [22]. These results confirm the interpretation that the CLW was excited in the Kiel experiment [22].

We now revisit the data from the Iowa experiment on transverse waves in a two-dimensional hexagonal lattice [23,32]. The dispersion relation for acoustic waves without damping (that is, $\nu = 0$), obtained earlier in [11], was used in [23] to fit the experimental data on $k_r(\omega)$. In the experiment, $k_i(\omega)$ is also measured but a theoretical fit was not attempted because the results given in [11] did not include the effect of damping. If the dispersion relation is of the form $k = k_0(\omega)$ for $\nu = 0$, we can obtain perturbative corrections to this dispersion relation for small damping by writing

$$k_r + ik_i \approx k_0(\omega + i\nu/2) \approx k_0(\omega) + \left.\frac{\partial k_0}{\partial \omega}\right|_{\nu=0}\left(\frac{i\nu}{2}\right) + O(\nu^2). \quad (7)$$

Equating the real and imaginary parts of Eq. (7), we obtain $k_r \approx k_0(\omega)$ and $k_i \approx (\nu/2)(\partial k_0/\partial\omega)_{\nu=0} = \nu/(2V_{g0})$, where V_{g0} is the group speed in the absence of damping. Clearly, while the correction to k_r due to small ν is of order ν^2, the correction to k_i is of order ν. This explains why, if the damping is weak, the dispersion relation for zero damping can be used to fit $k_r(\omega)$, but such a fit cannot detect deviations from theoretical predictions of order ν, reflected in $k_i(\omega)$.

In Fig. 3, for experimentally relevant parameters [23,32], we show the fit for the theoretical dispersion relation including the effect of damping for two different values of the angle θ. On the basis of this analysis, taking into account the size of the error bars, it appears that the waves excited in the Iowa experiment correspond approximately to $\theta \approx 0$.

One of the important features of longitudinal as well as transverse modes in the lattices is "negative dispersion":

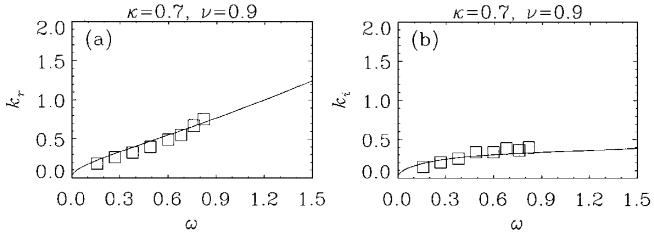

FIG. 2. The dispersion relation for the compressional lattice wave (solid line) is compared with experimental data (squares) from [22].

as k increases beyond the point where the group speed of the wave vanishes and standing waves can form, we eventually enter a regime of k where the group speed becomes negative. From Eq. (7) as well as Fig. 3, it is evident that, when the dispersion relation $k_r = k_r(\omega)$ approaches the limit $V_{g0} \to 0$, k_i increases sharply. The mode is then heavily damped in space, and it is difficult to observe any negative dispersion. This is actually seen in [23], where it appears to be difficult to push the experiment beyond a certain value of the real frequency ω.

In summary, we have presented a unified theoretical treatment of longitudinal (or compressional) and transverse waves in Yukawa crystals formed in a dusty plasma. We make theoretical predictions on two-dimensional hexagonal monolayers as well as three-dimensional bcc and fcc crystals. While the acoustic-wave limit of the relevant dispersion relations is isotropic in k, significant anisotropy develops at nonzero values of k. We have found that the κ dependence of the compressional waves depends sensitively on the dimensionality of the crystal. We have considered the effects of damping and compared theoretical dispersion relations quantitatively with experimental data [22,23]. We have also obtained other general dispersion relations that can be tested by future experiments. In particular, the change in the magnitude of the phase velocities of the waves as the wave number vector is oriented in different directions of the first Brillouin zone can be easily tested by changing the direction of laser manipulation. Because the damping depends inversely on the group speed (when damping is weak), the wave with the larger group speed will suffer less damping than that with the smaller group speed.

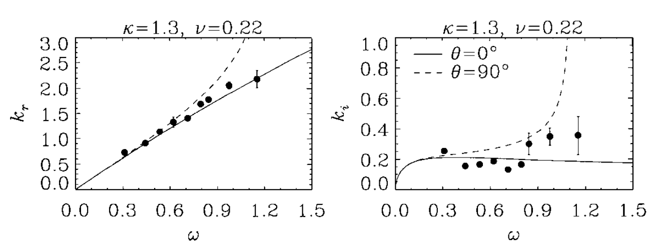

FIG. 3. The dispersion relation for the transverse lattice waves (solid line for $\theta = 0$, dashed line for $\theta = \pi/2$) is compared with experimental data from [23].

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