Grazing and Border-Collision in Piecewise-Smooth Systems: A Unified Analytical Framework

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A comprehensive derivation is presented of normal form maps for grazing bifurcations in piecewise smooth models of physical processes. This links grazings with border-collisions in nonsmooth maps. Contrary to previous literature, piecewise linear maps correspond only to nonsmooth discontinuity boundaries. All other maps have either square-root or (3/2)-type singularities.

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Many physical systems are characterized by the occurrence of nonsmooth events. Examples abound, including vibroimpacting mechanics and collision dynamics [1], switching electronic circuits [2], stick-slip motion [3], many physiological systems [4], and more generally any hybrid dynamical process involving discrete events. Such phenomena are often modeled by sets of piecewise-smooth (PWS) ordinary differential equations (ODEs), which are smooth in regions S_i of phase space, smoothness being lost as trajectories cross region boundaries $\Sigma_{i,j}$; see Fig. 1. Such models can exhibit rich bifurcation phenomena unique to their nonsmooth character, including so-called *grazing,* which occurs when a trajectory hits a boundary set $\Sigma_{i,j}$ tangentially. Grazing events are known to lead to a multitude of complex dynamical transitions, such as period-adding cascades and sudden transitions to a chaotic attractor, which have been observed both analytically [5,6] and experimentally [2,3,7]. However, the general theory is incomplete.

To develop a predictive tool for analyzing the observed dynamics caused by grazing bifurcations, one must construct appropriate *normal form* maps local to the grazing point. In the literature dealing with bifurcations of nonsmooth systems (e.g., [8]), it is often conjectured that such mappings are piecewise linear if the piecewise-smooth vector field is continuous across the boundaries. If this conjecture were true then their bifurcations could be studied by using the theory of C bifurcations [8] or border-collisions [9]. In contrast, if the system states are discontinuous, such as for a restitution law in impact oscillators, then the maps are known to have a square-root singularity [5,10]. It remains to be proved whether grazing in a PWS system with a continuous vector field leads to a piecewise linear map in general. Our analysis presented here indicates that this is often not the case [see also [6] for hypotheses that lead to a (3/2)-type map].

In this Letter, we propose a unified analytical framework for studying the local dynamics near grazing of general PWS systems. We establish a clear relationship between the continuity properties of the vector field at the grazing point and the functional form of the local map associated with it. We find that if grazing occurs with a smooth boundary [see Fig. 1(a)] the local map is indeed piecewise smooth but *never* piecewise linear. In contrast, if the boundary is itself nonsmooth and grazing takes place at a corner-type singularity where the vector field is discontinuous [see Fig. 1(b)], then the mapping is piecewise linear. We term this event a *corner-collision* bifurcation and we claim that this implies a border-collision of the corresponding local map.

These findings have immediate theoretical and experimental relevance for understanding phenomena in physical systems characterized by transitions between different smooth functional forms in macroscopic time scales. According to the nature of the system under investigation, we show that grazing events yield bifurcation scenarios which can be classified using different local maps. The overall results are summarized in Table I. Note that while there exist classification strategies for bifurcations in piecewise linear or square-root maps [9,10], the dynamics of maps with (3/2)-type singularities have not been fully analyzed.

The analytical framework we propose uses formal power series expansions and asymptotics which together give a synthetic analytical description of the grazing normal form map for a generic PWS system. We begin by assuming that sufficiently close to the grazing or corner-collision point, the phase space region under consideration is divided into two regions S_1 and S_2 by some boundary, Σ (see Fig. 1). This comprises either a smooth manifold [Fig. 1(a)] or a triangular wedge when projected onto a general plane [Fig. 1(b)]. In the former case, the *discontinuity boundary* is described by the zero set of a smooth codimension-one surface $H(x) = 0$. In the latter, the wedge is described instead by two smooth codimension-one surfaces Σ_1 and

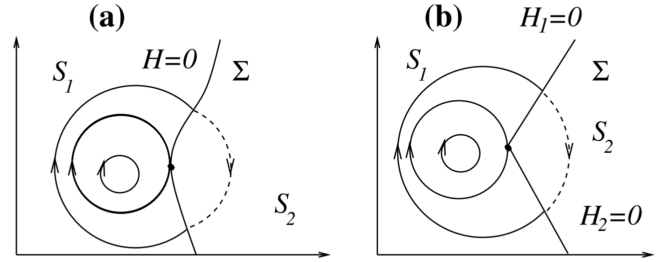

FIG. 1. Two-dimensional sketch graphs of (a) grazing and (b) corner-collision bifurcations.

TABLE I. Relationship between the properties of the system at the grazing point and the type of singularity in the corresponding local map.

System at grazing pt.		Map singularity
Nonsmooth boundary		Piecewise linear
Smooth boundary:		
F	Discontinuity in	
	x	Square-root [10]
Bounded	F	Square-root
C^0	F_x	(3/2)-type
C^1	F_{xx}	(3/2)-type

Σ_2 which are given by the zero sets of differentiable functions $H_1(x)$ and $H_2(x)$. These sets are supposed to intersect along a smooth codimension-two surface C (the corner) at a nonzero angle, i.e., $\nabla H_1 \times \nabla H_2 \neq 0$. In either case, the system near grazing can be described by the PWS ODE:

$$\dot{x} := F(x) = \begin{cases} F_1(x), & \text{if } x \in S_1\,, \\ F_2(x), & \text{if } x \in S_2\,, \end{cases} \qquad (1)$$

where $x \in \mathbb{R}^n$, F_1, F_2: $\mathbb{R}^n \mapsto \mathbb{R}^n$ are supposed to be sufficiently smooth and defined over the entire local region under consideration. For the sake of simplicity we further assume that the surfaces defined by the zero sets of $H(x)$, $H_1(x)$, and $H_2(x)$ are flat up to a sufficiently high order. Note that this may be assumed without loss of generality by making an appropriate sequence of near-identity transformations [11].

We say that a grazing occurs when a trajectory intersects a smooth boundary Σ tangentially. Without loss of generality this can be assumed to occur at the point $x = 0$ at which we further require that (a) $H^0 = 0$, (b) $\nabla H^0 \neq 0$, (c) $\langle \nabla H^0, F_i^0 \rangle = 0$, and (d) $\langle \nabla H^0, F_{ix}^0 F_i^0 \rangle > 0$. Here a superscript 0 represents a quantity evaluated at the grazing point $x = 0$. In contrast, if the discontinuity boundary is nonsmooth at $x = 0$, then a *corner-collision* bifurcation is said to occur under similar generic hypotheses, when the trajectory intersects Σ at this point. In both cases, we assume that Σ is never simultaneously attracting from regions S_1 and S_2, so that so-called Filippov solutions (or sliding modes) cannot exist. This final assumption can be similarly expressed by appropriate inequalities which we omit for brevity. (For the case of grazing in the presence of sliding motion, a complete analysis by di Bernardo, Kowalczyk, and Nordmark will be written up elsewhere.)

To perform the analysis, we make use of the concept of *discontinuity mapping* (DM); see [6]. This is the local map that describes the correction that must be made to the global Poincaré map from surfaces in S_1 in order to describe trajectories that pass through region S_2 close to $x = 0$. The DM is derived analytically by considering ε perturbations of a grazing or corner-colliding trajectory in the presence of the discontinuity boundary (see Fig. 2) by considering Taylor expansions of the flows Φ_1 and Φ_2 defined by

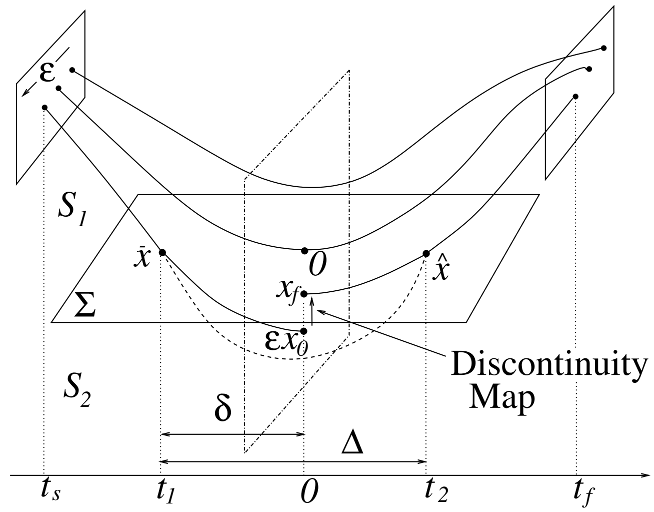

FIG. 2. Local analysis of grazing. A sketch graph of the three-dimensional case.

$$\frac{\partial \Phi_i}{\partial t} = F_i[\Phi_i(x,t)], \qquad \Phi_i(x,0) = x\,,$$

$$i = \begin{cases} 1 & \text{if } x \in S_1\,, \\ 2 & \text{if } x \in S_2\,. \end{cases}$$

As the vector fields are smooth, the flows $\Phi_i(x,t)$ can be expanded in Taylor series about the grazing point $(0,0)$:

$$\begin{aligned} \Phi_i(x,t) = {} & x + F_i^0 t + a_i t^2 + b_i x t + c_i t^3 + d_i x^2 t \\ & + e_i x t^2 + f_i t^4 + g_i x^3 t + h_i x^2 t^2 \\ & + j_i x t^3 + O(5)\,, \end{aligned} \qquad (2)$$

where a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, j_i are the matrix and tensor coefficients of the expansion and $O(5)$ is a shorthand for terms of order at least 5.

Consider first the case of a grazing bifurcation and let $x_g(t) = \Phi_1(0,t)$ be the trajectory which grazes the boundary at $x = 0$ when $t = 0$. Now, consider perturbations to x_g of size ε such that, for some unit vector x_0, along the trajectory $x(t) = \Phi_1(\varepsilon x_0, t)$ there exists some $t_1 = -\delta < 0$ at which the perturbed trajectory, $x(t)$, crosses Σ at $x = \bar{x}$ passing from S_1 into S_2. It is possible to work within a Poincaré section so that this condition is true if $\langle \nabla H, x_0 \rangle < 0$ (see [11]). The analysis can be split into three stages: motion in S_1 before the first crossing of the switching manifold, motion in S_2, and finally motion after the second crossing of Σ from S_2 to S_1.

The first step is to use the Taylor expansions (2) in order to derive an asymptotic expression for δ. In so doing, we find a unique positive solution for δ which can be expressed as an asymptotic expansion in $\sqrt{\varepsilon}$ of the form $\delta = \gamma_1 \varepsilon^{\frac{1}{2}} + \gamma_2 \varepsilon + \gamma_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$, where the coefficients γ_1, γ_2, γ_3 are expressed solely in terms of $F_{1,2}$ and their derivatives evaluated at the grazing point. Similarly, knowing δ, one can derive an estimate for $\bar{x}$; that is, $\bar{x} = \chi_1 \varepsilon^{\frac{1}{2}} + \chi_2 \varepsilon + \chi_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$.

In the second stage, the motion evolves on the other side of the boundary until after some time $t_2 = \Delta > 0$,

Σ is crossed again at $x = \hat{x}$. Here, we have $H(\hat{x}) = \Phi_2(\bar{x}, \Delta) = 0$. Using the Taylor series expansion of Φ_2 about the grazing point and the quantities computed in the previous stage, Δ can now be obtained as an asymptotic expansion in ε. Ignoring the trivial solution $\Delta = 0$, we get $\Delta = \nu_1 \varepsilon^{\frac{1}{2}} + \nu_2 \varepsilon + \nu_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$, where again the coefficients can be expressed in terms of the vector field and its derivatives evaluated at the grazing point.

In order to finally arrive at the DM, we proceed through the third stage as follows. We solve from the point $\hat{x} = \Phi_2(\bar{x}, \Delta)$, backwards in time through a time $-t_2$ using flow Φ_1 until we hit the Poincaré section containing the initial point εx_0. Here, we present the case of relevance to a periodically forced nonautonomous system where the appropriate Poincaré section is defined stroboscopically by $t = 0$. The more general case of autonomous systems can be treated similarly but leads to an algebraically more cumbersome expression. The discontinuity mapping is then the map from the initial point εx_0 to the final point $x_f = \Phi_1(\hat{x}, -t_2)$, where for the zero-time Poincaré section $t_2 = \Delta - \delta$. Using the asymptotic expansions for δ, $\bar{x}$, Δ and the expansion for $\hat{x} = \Phi_2(\bar{x}, \Delta)$ we can then systematically express x_f as a Taylor series in $\sqrt{\varepsilon}$.

Consider first the case of discontinuity of the vector field at the grazing point $F_1^0 \neq F_2^0$. Here we find the leading order term in x_f is $O(\varepsilon^{\frac{1}{2}})$. Specifically, we have $x_f = (F_2^0 - F_1^0)\nu_1 \varepsilon^{\frac{1}{2}} + O(\varepsilon)$. (Note that a square-root singularity is also observed in the case where F has a δ-function discontinuity at $x = 0$ [10].)

Next, suppose instead that the vector field is continuous at the grazing point but has discontinuous Jacobian, i.e., $F_1^0 = F_2^0$ and $F_{1x}^0 \neq F_{2x}^0$. Then the $O(\varepsilon^{\frac{1}{2}})$ contribution to x_f vanishes and it is possible to show that the $O(\varepsilon)$ contribution to the DM is just the identity. Hence the leading-order nontrivial term is $O(\varepsilon^{\frac{3}{2}})$. This is also true if the Jacobian is continuous but the Hessian is not, i.e., $F_1^0 = F_2^0$, $F_{1x}^0 = F_{2x}^0$, but $F_{1xx}^0 \neq F_{2xx}^0$.

Lengthy algebraic manipulations [11] allow the analytical derivation of the leading-order part of the discontinuity mapping in the two cases treated above. This in turn allows explicit expressions for the normal forms associated with hyperbolic periodic orbits undergoing grazing in general n-dimensional PWS systems. Specifically, we present here formulas for the case when the grazing trajectory is part of a hyperbolic mT-periodic orbit $p(t)$ of a T-periodically forced system. That is (omitting the superscript 0 on each quantity involving F and H):

(I) If the vector field is discontinuous at grazing, we have

$$x \mapsto \begin{cases} Nx + M\mu, & \text{if } \langle \nabla H, x \rangle > 0, \\ N\mathbf{w}\sqrt{|\langle \nabla H, x \rangle|} + M\mu + \text{h.o.t.} & \text{if } \langle \nabla H, x \rangle < 0, \end{cases}$$

where

$$\mathbf{w} = 2(F_2 - F_1)\frac{\langle \nabla H, F_{2x}F_1 \rangle}{\langle \nabla H, F_{2x}F_2 \rangle}\left(\frac{2}{\langle \nabla H, F_{1x}F_1 \rangle}\right)^{\frac{1}{2}};$$

grazing occurs at $\mu = 0$, and N and M are the linear parts of the Poincaré map calculated using flow Φ_1 alone.

(II) If the vector field is continuous at $x = 0$, i.e, $F_1 = F_2 := F$, but has discontinuous Jacobian (or Hessian):

$$x \mapsto \begin{cases} Nx + M\mu, & \text{if } \langle \nabla H, x \rangle > 0, \\ N[x + \mathbf{v}_1(|\langle \nabla H, x \rangle|)^{\frac{3}{2}} + V_2 x(|\langle \nabla H, x \rangle|)^{\frac{1}{2}} + \mathbf{v}_3 \langle \nabla H, F_{2x}x \rangle (|\langle \nabla H, x \rangle|)^{\frac{1}{2}}] + M\mu & \text{if } \langle \nabla H, x \rangle < 0, \end{cases}$$

where

$$\mathbf{v}_1 = \frac{1}{\langle \nabla H, F_{1x}F_1 \rangle^{\frac{3}{2}}}\Bigg\{\frac{2}{3}(F_{2xx} - F_{1xx})F^2 + 2F_{2x}F_{1x}F - \frac{2}{3}[(F_{1x})^2 + 2(F_{2x})^2]F - \frac{2}{\langle \nabla H, F_{2x}F_2 \rangle}(F_{2x} - F_{1x})F \cdot \left[\frac{2}{3}\langle \nabla H, [F_{2xx}F_2^2 + (F_{2x})^2 F_2] \rangle + \langle \nabla H, [F_{2x}F_{1x} - 2(F_{2x})^2]F \rangle + \langle \nabla H, F_{2xx}F^2 \rangle\right]\Bigg\}$$

$$V_2 = \frac{2}{\sqrt{\langle \nabla H, F_{1x}F_1 \rangle}}(F_{2x} - F_{1x}) \qquad \mathbf{v}_3 = \frac{2}{\langle \nabla H, F_{2x}F_2 \rangle \sqrt{\langle \nabla H, F_{1x}F_1 \rangle}}(F_{2x}F_2 - F_{1x}F_1).$$

So, contrary to what has been assumed in the literature, our results rule out the possibility of piecewise linear maps associated to grazing events involving a smooth discontinuity boundary. However, experiments on a certain class of electronic circuits, so-called dc/dc power converters, indicate that piecewise linear maps can indeed be observed at a corner-collision point [12]. To prove that this is true, we must adapt the above analytical framework in order to take into account the geometry of the corner. For brevity, we consider only the so-called external corner-collision depicted in Fig. 1(b); the internal case can be studied similarly [13].

With careful consideration of higher-order terms, the discontinuity mapping can again be constructed by expanding the system flows about the corner-collision point. Now, though, the linear terms in the expansions can be shown to be sufficient to completely describe the local dynamics near the corner. This can be explained heuristically that in the grazing case a locally parabolic tangency occurs while at a corner-collision the time spent "inside" the corner varies linearly with ε. Specifically, it is possible to show that, taking into account trajectories that do not cross the wedge, the DM is simply

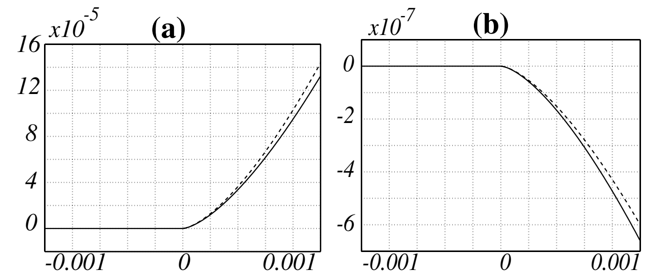

FIG. 3. Theoretical prediction (dashed line) and numerical simulation (solid line) of the change to the local behavior of Eq. (3) near grazing, when (a) the stiffness or (b) the damping is varied across Σ. $x_f - \varepsilon x_0$ is plotted against ε. The parameters are set to be (a) $k_1 = 1$, $k_2 = 2$, $\zeta_1 = \zeta_2 = 0.1$; (b) $k_1 = k_2 = 2$, $\zeta_1 = 1$, $\zeta_2 = 0.1$; while $\beta_1 = \beta_2 = 1$.

$$x \mapsto \begin{cases} x, & \text{if noncrossing,} \\ x + (F_1^0 - F_2^0)\langle \alpha, x\rangle + o(|x|) & \text{if crossing,} \end{cases}$$

where $\alpha = J_2 - \langle J_2, F_1^0\rangle J_1$ with $J_i = \nabla H_i^0 / \langle \nabla H_i^0, F_i^0\rangle$.

Example 1.—As a simple representative example we consider the case of one-degree-of-freedom forced damped harmonic motion in a medium whose characteristics change at $x = 0$:

$$\ddot{x} + \zeta_i \dot{x} + k_i^2 x = \beta_i \cos(t), \qquad i = \begin{cases} 1 & \text{if } x > 0, \\ 2 & \text{if } x < 0. \end{cases} \tag{3}$$

In this case, the boundary between the two regions of smooth dynamics S_1 and S_2 is the line $\Sigma := \{x = 0\}$. The change in the medium is modeled by a variation of the linear stiffness ($k_1 \neq k_2$), damping coefficient ($\zeta_1 \neq \zeta_2$) or amplitude of the forcing term ($\beta_1 \neq \beta_2$). Recasting (3) as a set of first-order ODEs, it is possible to see that the vector field is continuous but has discontinuous Jacobian if $\beta_1 = \beta_2$ and $k_1 \neq k_2$ or $\zeta_1 \neq \zeta_2$, while it is discontinuous if $\beta_1 \neq \beta_2$. Therefore, the above analysis predicts that the local behavior near grazing is described by a map with a square-root singularity in the latter case or a $(3/2)$-type singularity in the former. This agrees perfectly with the numerics depicted in Figs. 3 and 4.

Example 2.—To illustrate the corner-collision case, we now take system (3) but suppose that the discontinuity boundary is now a nonsmooth wedge defined by the zero sets of $H_{1,2}(x) = x \mp \dot{x}$ (as, for instance, in the so-called trilinear oscillator [1]). In this case, the analysis gives local dynamics described by a piecewise linear map, which is indeed confirmed by numerical results [Fig. 4(b)].

In conclusion, many physical systems undergo periodic behavior which grazes with some discontinuity caused by a switching or impacting event. We have presented an analytical framework for modeling such events and hence explaining the observed dynamical consequences. The key step is to reduce the dynamics to its essentials: by deriving normal form maps relying only on assumptions about the discontinuity at the graze. In so doing, we have provided the first consistent link between the concepts of grazing bifurcations in continuous-time systems and border collisions in PWS maps. We have shown that a special case of "corner-collision" leads to piecewise linear maps, whereas all other cases lead to maps with either a square-root or a $(3/2)$-type singularity as given in Table I. The complete classification of the dynamics associated with these maps is the subject of ongoing research.

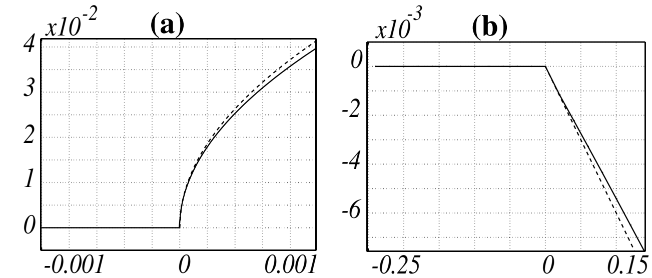

FIG. 4. Comparison between theory and numerics defined similarly to Fig. 3 for (a) Eq. (3) near grazing, when the amplitude of the forcing term is varied across Σ ($k_1 = k_2 = 2$, $\zeta_1 = \zeta_2 = 0.1$, $\beta_1 = 1$, $\beta_2 = 2$); (b) for a corner-collision in the modified Eq. (3) (trilinear oscillator) with $k_1 = k_2 = \sqrt{5}$, $\zeta_1 = \zeta_2 = 0.55$, $\beta_1 = 4.04$, $\beta_2 = 6.04$.

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