Stagger-and-Step Method: Detecting and Computing Chaotic Saddles in Higher Dimensions

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Chaotic transients occur in many experiments including those in fluids, in simulations of the plane Couette flow, and in coupled map lattices. These transients are caused by the presence of chaotic saddles, and they are a common phenomenon in higher dimensional dynamical systems. For many physical systems, chaotic saddles have a big impact on laboratory measurements, but there has been no way to observe these chaotic saddles directly. We present the first general method to locate and visualize chaotic saddles in higher dimensions.

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Chaotic transients are a common phenomenon in higher dimensional dynamical systems and were observed in the Lorenz system [1], in experiments on fluids [2,3], in many low-dimensional systems [4,5], in spatiotemporal chaotic systems [6], in simulations of the plane Couette flow (a shear flow between two parallel walls) [7–9], and in coupled map lattices [10]. Generally, these transients are caused by the presence of a chaotic saddle [11,12]. We first describe, in some detail, a particular situation to illustrate how chaotic saddles can strongly affect observed experiments. Transition to turbulence in shear flows is still not fully understood. The numerical model for shear flow, studied in [8], has 19 degrees of freedom. For low Reynolds numbers, it has a single stable stationary state, laminar flow. Laminar flow is stable for all Reynolds numbers. Reference [8] reports that, at a certain Reynolds number, new stationary saddles appear and presumably a chaotic saddle. The dynamics of perturbations outside the domain of attraction of the laminar profile is very complicated. They argue that phase space trajectories (transient turbulent excitations) will eventually escape the tangle and find their way to the laminar profile. Thus, the turbulent structures seen are transient [8]. The full numerical simulations in [7] of the plane Couette flow and the 19 degree of freedom model in [8] have rather similar behavior. They report that the distribution of lifetimes as a function of perturbation amplitude and Reynolds number has a self-similar fractal structure [7–9]. This suggests that the long lifetimes arise for those initial perturbations which come close in phase space to a chaotic saddle, and it suggests that arbitrarily large lifetimes should be possible and the "turbulent" state would be supported by a chaotic saddle rather than an attractor. The long-lived transients in pipe flow [3] imitate a permanent turbulent state, making it extremely difficult to determine the precise transition to sustained turbulence (if it exists). These experiments all strongly suggest the existence of a chaotic saddle. But the chaotic saddles have not yet been directly observed except in the simplest fluid models [1].

The purpose of this Letter is to present a simple, general method for locating chaotic saddles that are responsible for the transient chaotic behavior. Our model examples are still much simpler than the shear flow models above, which would require major computer resources for investigation. There are a small number of methods for detecting chaotic saddles. The "Sprinkle method" [5] works well for many systems in two dimensions, but not in higher dimensions. It finds only points within 10^{-3} of the chaotic saddle and cannot be used for computing Lyapunov exponents. The "PIM-triple method" [11] detects and computes chaotic saddles (even in high-dimensional phase space) provided the unstable dimension is one. This method generates trajectories which are very close to the stable set (e.g., within 10^{-8}), and Lyapunov exponents can be computed. It has been used in chaotic scattering, in open hydrodynamical flows, for communicating with chaos, in a modulated class-B laser [13]. In the spirit of the PIM-triple method, Jánosi *et al.* [14] developed a method for reconstructing chaotic saddles from experimental time series as in an NMR-laser experiment [15]. However, if the chaotic saddle is k-dimensionally unstable for $k > 1$, the PIM-triple method will typically fail. In a recent breakthrough, Moresco and Dawson [16] created a new, complicated "PIM-simplex method," showing it is sometimes possible to deal with chaotic saddles having unstable dimension two or higher, but the applicability of the method is quite limited. The transition to turbulence in shear flows, as discussed above, apparently have chaotic transients which are unstable in several dimensions. Therefore, it is important to have a general method that detects and computes such chaotic saddles.

In this Letter, we report a general method for finding numerically chaotic trajectories on chaotic saddles, a method that works even if the unstable dimension is two or higher. We call this method the "Stagger-and-Step method" and it can be applied to flows (differential equations) or maps. Our main result is that the Stagger-and-Step method generates a long numerical trajectory on the chaotic saddle. The method works fine for four-dimensional systems such as the coupled Hénon maps and kicked double rotor map with several parameter values, and we expect it to work for a wide variety of higher dimensional problems. We present the Stagger-and-Step method for maps, but the method works the same for differential equations. Just calculate the escape time for the ordinary differential equation. (To do this, consider the map generated by the numerical differential equation solver which takes the system from a point in phase space at time t_n to a point at time $t_{n+1} = t_n + \Delta t_n$ and apply the method.) Let F be a continuous map from the phase space $\mathbb{R}^d$ into itself ($d \geq 2$). Let R be a "transient region" (that is, a region in $\mathbb{R}^d$ that contains no attractor). We investigate the very special trajectories that remain in R for all positive time. The *escape time* $T(\mathbf{x})$ of $\mathbf{x}$ from R is the minimum $n \geq 0$ for which the nth iterate of $\mathbf{x}$ is not in R; write $T(\mathbf{x}) = \infty$ if all forward iterates of $\mathbf{x}$ are in R. Let C be the largest invariant set of F in R; that is $F(C) = C$, so $T(\mathbf{x}) = \infty$ for $\mathbf{x} \in C$. Assume that C is nonempty and is "unstable," i.e., the escape time of almost every (in the sense of Lebesgue measure) point $\mathbf{x} \in R$ is finite. The *stable set,* S_C, of C is the collection of all points $\mathbf{x} \in R$ such that $T(\mathbf{x}) = \infty$. If the escape time $T(\mathbf{x})$ of a point $\mathbf{x}$ is high, then the point $\mathbf{x}$ is close to S_C. The collection of points A_t with escape time at least t, for t large, is a small neighborhood of S_C, and as t increases $A_t \rightarrow S_C$.

Our new *Stagger method* finds points on (or extremely close to) S_C by finding points with high escape times. A *stagger* is a perturbation $\mathbf{r}$ of a point $\mathbf{x}$ resulting in $\mathbf{x} + \mathbf{r}$ such that $T(\mathbf{x} + \mathbf{r}) > T(\mathbf{x})$. This method creates "stagger trajectories," that is, sequences $\{\mathbf{x}_n\}_{n\geq 0}$ of the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{r}_n\,, \tag{1}$$

where $\mathbf{r}_n$ is a stagger, so $T(\mathbf{x}_{n+1}) > T(\mathbf{x}_n)$. The process stops as soon as $T(\mathbf{x}_{n+1}) \geq T^*$, for some specified $T^* > 0$. To implement this method, specify some (large) $\delta > 0$, and, for each n, we repeatedly choose random points $\mathbf{r} \in \mathbb{R}^d$ with $\|\mathbf{r}\| \leq \delta$ (using some specified distribution) until we find one with $T(\mathbf{x}_n + \mathbf{r}) > T(\mathbf{x}_n)$; we then set $\mathbf{r}_n = \mathbf{r}$. In some situations, no such $\mathbf{r}_n$ exists if δ is too small. The applicability of this method is due to our choice of distribution used for choosing $\mathbf{r}$. Figure 1(a) shows the usefulness of our choice of distribution used for choosing $\mathbf{r}$. We argue that, if $\mathbf{r}$ was chosen from a uniform distribution (with $\|\mathbf{r}\| \leq \delta$), then the fraction of perturbations which are staggers would go to zero exponentially fast as the escape time $t \rightarrow \infty$. To see this, assume that, at $t = 0$, one selects N_0 initial points (for N_0 large) from a uniform distribution on R. Evolve these N_0 points under the dynamics. For $t \geq 0$, let N_t denote the number of trajectories that are still in R at time t. Typically, N_t decays exponentially as time t increases [17]; see also the caption of Fig. 1(a). When using the "Exponential Stagger Distribution" for choosing $\mathbf{r}$, we find that the fraction of staggers decreases much slower as $t \rightarrow \infty$, and typically the probability that $[T(\mathbf{x}_n + \mathbf{r}) > T(\mathbf{x}_n)]$ is much higher than when using the uniform distribution as escape time increases, so the computation time to get $\mathbf{x}_{n+1}$ from $\mathbf{x}_n$ in Eq. (1) is considerably lower. We now state the distribution we use in our methods.

Exponential Stagger Distribution.—Let $\delta > 0$, and let a be such that $10^{-a} = \delta$. Let s be a uniformly distributed random number between a and 15 (when we are computing with 15-digit precision). Choose a random unit direction vector $\mathbf{u} \in \mathbb{R}^d$ and define $\mathbf{r} = 10^{-s}\mathbf{u}$. (The vector $\mathbf{u}$ is chosen from a uniform distribution on the set of unit vectors.)

To illustrate the effectiveness of the Stagger method for finding points having high escape times, we apply it to two coupled Hénon maps. Let the map F from the phase space $\mathbb{R}^4$ into itself be defined by

$$F(x,y,u,v) = (A - x^2 + By + k(x - u), x, C - u^2 + Dv + k(u - x), u)\,, \tag{2}$$

where $0 \leq k \leq 1$. We choose $A = 3$, $B = 0.3$, $C = 5$, $D = 0.3$, and $k = 0.4$. Select the transient region R to be $(-4,4) \times (-4,4) \times (-4,4) \times (-4,4)$, and let δ be the length of the diagonal of R, so $\delta = 16$. Numerical experiments show that for a stagger trajectory to proceed from escape time 5 to escape time 31 requires on the average about 50 (31 − 5) = 1300 choices of $\mathbf{r}$; that is, about 50 choices of $\mathbf{r}$ were needed before a (successful) stagger occurs, i.e., finding an $\mathbf{r}$ which increases the escape time. Figure 1(a) shows the probability of such a success is largely independent of escape time [18], for times in the interval [5,30], after which it drops precipitously. Figure 1(a) shows the results from the talley equivalent of examining 90 000 stagger trajectories.

We now present the Stagger-and-Step method that allows us to compute a trajectory on a chaotic saddle. Select any initial condition in R, choose any $\delta > 0$, and choose $T^* > 0$ (the minimal required escape time). Using a stagger trajectory (with large δ), find a point $\mathbf{x}_0$ with $T(\mathbf{x}_0) \geq T^*$. Now choose a small $\delta \approx 10^{-10}$. The *Stagger-and-Step method* generates "Stagger-and-Step" trajectories $\{\mathbf{x}_n\}_{n\geq 0}$ of the form

$$\mathbf{x}_{n+1} = \begin{cases} F(\mathbf{x}_n) & \text{if } T(\mathbf{x}_n) > T^* \quad \text{(a step)} \\ F(\mathbf{x}_n + \mathbf{r}_n) & \text{if } T(\mathbf{x}_n) \leq T^* \quad (\mathbf{r}_n \text{ is a stagger})\,, \end{cases} \tag{3}$$

where $\|\mathbf{r}_n\| \leq \delta$ and $T(\mathbf{x}_n + \mathbf{r}_n) > T(\mathbf{x}_n)$. Hence, if $\mathbf{x}_n$ has escape time $T(\mathbf{x}_n) > T^*$, then apply F. Note that $T(F(\mathbf{x}_n)) = T(\mathbf{x}_n) - 1$. If $T(\mathbf{x}_n) = T^*$, find a nearby point $\mathbf{x}_n + \mathbf{r}_n$ with a higher escape time and then apply F. (In some situations, no such $\mathbf{r}_n$ exists if δ is too small.) The perturbation $\mathbf{r}_n$ in Eq. (3) is obtained as in the stagger method (except δ is small). Any Stagger-and-Step trajectory $\{\mathbf{x}_n\}_{n\geq 0}$ satisfies (by construction) $\|F(\mathbf{x}_n) - \mathbf{x}_{n+1}\| < \delta$, i.e., $\{\mathbf{x}_n\}_{n\geq 0}$ is a δ-pseudo trajectory. Hence, $\{\mathbf{x}_n\}_{n\geq 0}$ is a numerical trajectory with numerical precision of the order of $\delta = 10^{-10}$, and

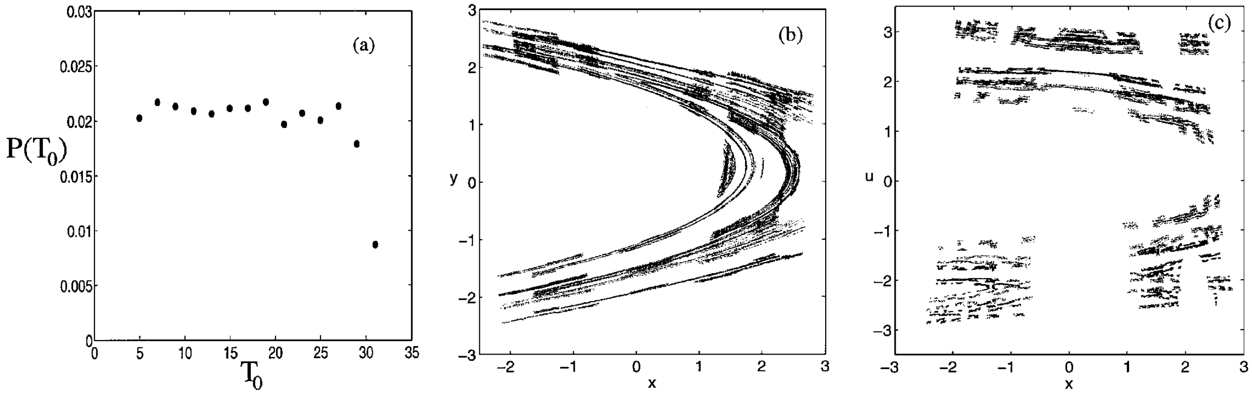

FIG. 1. The map $F(x, y, u, v) = (A - x^2 + By + k(x - u), x, C - u^2 + Dv + k(u - x), u)$, with $A = 3$, $B = 0.3$, $C = 5$, $D = 0.3$, and $k = 0.4$. (a) For a fractal set S in $\mathbb{R}^m$ (such as the stable set of a chaotic saddle), the volume of the ε-neighborhood of S (in a bounded region) scales like ε^{m-d}, where d is the box-counting dimension of S (McDonald *et al.* [21]). Hence, the probability of coming within ε of the set S is proportional to the volume of its ε-neighborhood, which is proportional to ε^{m-d}. $P(T_0)$ can be thought of as the probability that a perturbation $\mathbf{r}$ is a stagger when $\mathbf{r}$ is chosen using the Exponential Stagger Distribution. Since $P(T_0)$ is approximately constant on [5,29], one may select T^*, e.g., $T^* = (5 + 29)/2 = 17$, to be an admissible escape time. (b),(c) The chaotic saddle for the map F that is obtained by applying the Stagger-and-Step method using $\delta = 10^{-10}$ and $T^* = 30$. The figures were created using one Stagger-and-Step trajectory having 10^5 points. The Lyapunov exponents of this trajectory are $\lambda_1 \approx 1.33$, $\lambda_2 \approx 0.77$, $\lambda_3 \approx -1.97$, and $\lambda_4 \approx -2.54$.

typically it approximates the chaotic saddle after a few iterates. In other words, the Stagger-and-Step method generates a numerical trajectory on the chaotic saddle. Note that statistics (such as Lyapunov exponents) of such a trajectory can be computed directly.

To illustrate the Stagger-and-Step method, we first apply this method to two coupled Hénon maps. Let F be the map in Eq. (2) with the same parameter values and transient region as above. The maximal invariant set C is two-dimensionally unstable, and the *PIM-triple method fails to generate chaotic trajectories* on C. We choose $T^* = 30$. Having found an $\mathbf{x}_0$ with $T(\mathbf{x}_0) = 30$, we set $\delta = 10^{-10}$. Applying the Stagger-and-Step method results in a single numerical trajectory [shown in Fig. 1(b) after discarding the first 100 points]. This numerical trajectory is very close to the chaotic saddle, and its finite time Lyapunov exponents are $\lambda_1 \approx 1.33$, $\lambda_2 \approx 0.77$, $\lambda_3 \approx -1.97$, and $\lambda_4 \approx -2.54$. Numerical studies based on the Stagger method show the points $\mathbf{x}_n$ are in a δ-neighborhood of points with much higher escape time ($t = 30$), so it may be inferred the $\mathbf{x}_n$ are within 2δ of the maximal invariant set. We now examine an example involving the double rotor map to illustrate the Stagger-and-Step method. The double rotor, introduced in [19] and revised in [20], is composed of two thin, massless rods connected as shown in Fig. 2(a). We refer to [19,20] for the description of the double rotor. The differential equations that describe the kicked double rotor are presented in [19,20]. From these equations, they derive the "kicked double rotor map" that relates the angles (θ and φ) and velocities ($\dot{\theta}$ and $\dot{\varphi}$) immediately after the $(n + 1)$th kick with those immediately after the nth one [19]; see [20] for the equations of the double rotor map. This map is studied in [20] for $\nu_1 = \nu_2 = T = I = m_1 = m_2 = \lambda = 1$, $L = \frac{1}{2}\sqrt{2}$, and the parameter ρ is varied. We select the same parameter values, and we choose $\rho = 12$. Select the transient region R to be $(\pi/4, 5\pi/3) \times (0, 2\pi] \times (-14, 14) \times (-16, 16)$ and we choose $\delta = 1$. Note θ near 0 or 2π is not in R. The maximal invariant set C is two-dimensionally unstable, and *the PIM-triple method fails to generate chaotic trajectories* on C. We choose $T^* = 33$. First, the Stagger method was applied using $\delta = 1$ to find a point having escape time at least T^*. Having found an $\mathbf{x}_0$ with $T(\mathbf{x}_0) = 33$, we set $\delta = 10^{-10}$. Applying the Stagger-and-Step method results in a single numerical trajectory [shown in Fig. 2(b) after discarding the first 100 points]. This numerical trajectory is very close to the chaotic saddle; its finite time Lyapunov exponents are $\lambda_1 \approx 1.12$, $\lambda_2 \approx 0.33$, $\lambda_3 \approx -1.24$, and $\lambda_4 \approx -2.87$.

We now address the following question: What is a good choice for T^* so that a Stagger-and-Step trajectory is likely to yield an entire chaotic saddle on which F is transitive? We say that T^* is an *admissible escape time* if ($A1$) for every point $\mathbf{p} \in C$, there is some computer point $\mathbf{p}_c$ for which $\|\mathbf{p} - \mathbf{p}_c\| < \delta$ and $T(\mathbf{p}_c) \geq T^*$, and ($A2$) if $\mathbf{p}^c$ is a computer point with $T(\mathbf{p}^c) \geq T^*$, then $\mathbf{p}^c$ is within δ of some point in C. To determine T^* explicitly in an example, we consider piecewise linear maps. For $a > 0$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the well-studied symmetric tent map; that is, $f_a(x) = ax$ if $x \leq 0.5$ and $f_a(x) = a(1 - x)$ if $x \geq 0.5$. We assume that $a > 2$. Then there is a chaotic saddle. Select $(-1, 2)$ to be the transient region, and let $\delta = 10^{-8}$. For each value a, we can determine an interval of T^* that guarantees that a Stagger-and-Step trajectory remains in a δ neighborhood of the invariant Cantor set. For example, when $a = 3$, the admissible minimal escape times are $20 \leq T^* \leq 33$. [For $T^* < 20$ ($A1$) holds but ($A2$) does not; for $T^* > 33$ ($A1$) fails.] For $a = 20$, we get $8 \leq T^* \leq 12$. We now investigate the map $g_{a,b}$:

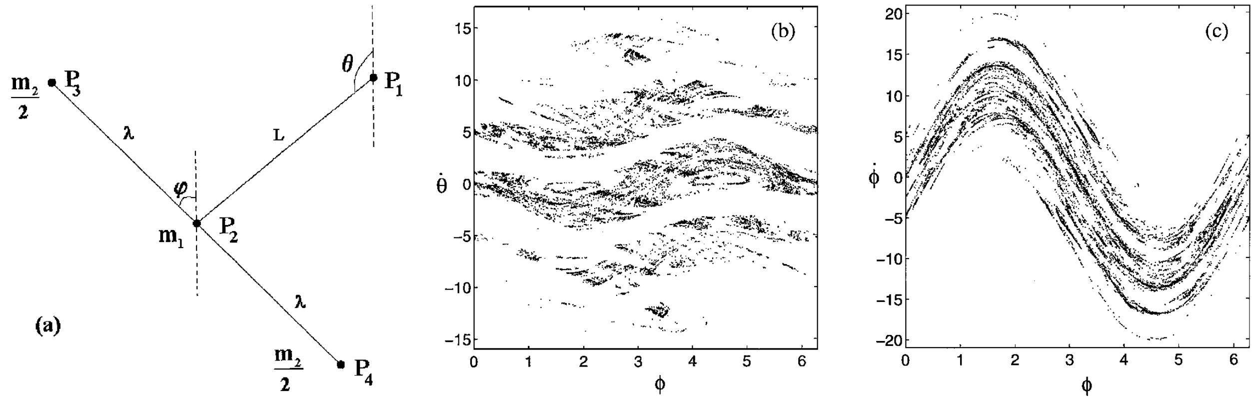

FIG. 2. (a) The double rotor. (b),(c) These figures are slices of a chaotic saddle for the double rotor map and were created using one stagger and step trajectory having 4 000 000 points ($T^* = 33$, $\delta = 10^{-10}$). The figures show two projections of approximately 10 000 points satisfying $|\theta - \pi| < 0.01$. The resulting Lyapunov exponents of this trajectory are $\lambda_1 \approx 1.12$, $\lambda_2 \approx 0.33$, $\lambda_3 \approx -1.24$, and $\lambda_4 \approx -2.87$.

$\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $g_{a,b}(x, y) = (f_a(x), f_b(y))$. We assume that computation uses fixed precision with machine $\varepsilon = 10^{-15}$; then it follows from the results above that the Stagger-and-Step method will not work for $a = 3$, $b = 20$, since the intervals of minimal escape time do not intersect. Indeed, no numerical method based on fixed escape times will be able to generate trajectories lying in a δ-neighborhood of the chaotic saddle. However, if one computes the Stagger-and-Step method in fixed precision with machine $\varepsilon = 10^{-25}$, then the Stagger-and-Step method generates a numerical trajectory on the chaotic saddle of the map $g_{a,b}$ for $a = 3,\ b = 20$.

In summary, we use the Stagger method to find points with high escape times, for example, to initialize the Stagger-and-Step method. The Stagger-and-Step method is the first general, but simple, method for finding trajectories on chaotic saddles, trajectories whose statistics (such as Lyapunov exponents) can be computed directly. In addition, it may be used for computing the dimension of a chaotic saddle. Because of limited computer precision, the method might fail if the ratio between two positive Lyapunov numbers of a system is too big, e.g., if two of the Lyapunov numbers are 100 and 1.01. We expect the Stagger-and-Step method to be a useful additional tool in analyzing the dynamics of fluids, lasers, and transition to turbulence in shear flows.

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