On the Significance of the Vector Potential Squared

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We consider the gauge potential A and argue that the minimum value of the volume integral of A^2 (in Euclidean space) may have physical meaning, particularly in connection with the existence of topological structures. A lattice simulation comparing compact and noncompact "photodynamics" shows a jump in this quantity at the phase transition, supporting this idea.

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Introduction.—Physical quantities should be gauge invariant. At first glance this might seem to imply that only expressions involving the fields (E and B in electromagnetism) and not the potentials (A) should appear in physically meaningful quantities, and in fact this is usually true. However, this logic can be misleading. A well known case in point is the loop integral $\oint \mathbf{A}\, dx$. Although only A, and not the fields, appears explicitly in this construction, it is so devised that it leads to a gauge invariant and indeed an interesting object.

We would like to point out the interest of another quantity constructed from A itself: the volume integral of $A^2(x)$. One may come upon this thought when considering the role of condensates in quantum field theory. Vacuum condensates have been a useful way to understand and characterize the dynamics of QCD and other field theories. The most famous example is perhaps the quark condensate:

$$\langle 0|\bar{q}q|0\rangle \neq 0\,, \tag{1}$$

where q stands for light u or d quarks. In the realistic case of negligibly small quark mass, a nonvanishing value of the quark condensate signals spontaneous breaking of chiral symmetry.

In the framework of the QCD sum rules [1] one also used the concept of the gluon condensate

$$\langle 0|\alpha_s(G^a_{\mu\nu})^2|0\rangle \neq 0\,. \tag{2}$$

Here the nonvanishing value of the condensate signifies not the breaking of a symmetry but rather the presence of nonperturbative fields in the vacuum.

This gluon condensate would appear to be the simplest quantity characterizing nonperturbative vacuum fields. It has dimension $d = 4$, leading one to assume that the leading nonperturbative corrections in the QCD sum rules at large external momentum Q are of order $\langle 0|\alpha_s(G^a_{\mu\nu})^2|0\rangle/Q^4$.

Now there is of course an even simpler candidate for a condensate, namely, just the square of the vector potential: A^2. This is of dimension $d = 2$. However, such expressions seem not to be allowed since they appear gauge noninvariant [2]; that is, one tends to think that physically meaningful quantities must involve only the fields and not the potentials and that an expression like A^2, involving only potentials, could not be meaningful. However, this is not necessarily true, as we now illustrate on the simple example of magnetostatics.

Magnetostatics.—Consider a situation with some magnetic field $\mathbf{B}$ present in space. There is a considerable amount of freedom in the choice of $\mathbf{A}$. However, since there is a nonzero magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, we know *some* nonzero $\mathbf{A}$ must be present; $\mathbf{A}$ cannot be zero everywhere. Now consider the volume integral of $A^2(x)$. It is a positive quantity and cannot be zero. It must then have some minimum value. Therefore of all the possible $\mathbf{A}$ configurations which yield the given $\mathbf{B}$ the one (or the ones) with the smallest integral of $\mathbf{A}^2$ has in a sense an invariant significance. We then examine the possible significance of the resulting quantity, the volume integral of $A^2(x)$ at its minimum value. We will call this $A^2_{\rm min}$.

The connection between the "minimum A^2" requirement and a more familiar gauge condition may be seen as follows. Suppose for a given field configuration that $\int \mathbf{A}^2\, d^3x$ is at its minimum value; then under a gauge transformation it is stationary. Considering $\mathbf{A} \rightarrow \mathbf{A} + \nabla\phi$ for infinitesimal ϕ we have $\int \mathbf{A}\nabla\phi\, d^3x = 0$ and integrating by parts

$$\int \phi \nabla \mathbf{A}\, d^3x \,+\, \text{surface terms} = 0\,. \tag{3}$$

Since ϕ is arbitrary we conclude that, up to the surface terms and the question of local minima in A^2, the minimum A^2 condition is equivalent to the familiar gauge condition

$$\nabla \mathbf{A} = 0\,. \tag{4}$$

Not surprisingly the minimum A^2 requirement is connected with that gauge condition which is invariant, i.e., makes no reference to any particular direction.

We emphasize, however, that our interest is focused not so much on the minimization of A^2 as a gauge condition as on the *value* of the quantity itself. This is somewhat analogous to the role of the action in classical and quantum mechanics. In classical mechanics one simply minimizes the action but is not particularly concerned with its actual value. When one comes to quantum mechanics, however, it is recognized that there is a significance to the value of the action itself.

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Furthermore, it appears that $A^2_{\min}$ is sensitive to, or measures in some way, the existence of nontrivial structures in the system under consideration. This is suggested by the comparison of two situations, both with no magnetic field. Let one be simple empty space with $\mathbf{B} = \mathbf{0}$, while the other has a nontrivial topology with the presence of a tube or string containing magnetic flux, like a "cosmic string" or a vortex in superconductivity. In the first case we have simply no A and so $A^2_{\min} = 0$. In the second case, due to the flux Φ in the tube or string

$$\oint \mathbf{A}\, dx = \int \mathbf{H} \cdot d\mathbf{s} \equiv \Phi\,, \tag{5}$$

and A cannot be zero in the surrounding space even though the magnetic field is absent. This example suggests that $A^2_{\min}$ can signal the presence of nontrivial topological structures.

The logical situation concerning $A^2_{\min}$ resembles somewhat that of the question of the energy of a particle in relativity. The energy of a particle is of course a frame dependent quantity. However, the minimum energy, which is the energy in the rest frame, has an invariant meaning, namely, the mass. In going to the rest frame of the particle we do make a certain choice of frame, but nevertheless the mass is an undeniably meaningful quantity [3].

Of course the mass also has an explicitly invariant expression, $m^2 = E^2 - P^2$, and the loop integral $\oint \mathbf{A}\, dx$ can, via Stokes theorem, be expressed in term of the fields. Analogously, is there an expression for $A^2_{\min}$ directly in terms of the fields?

Indeed there is the vector relation [4]

$$\int \mathbf{A}^2(x)\, d^3x = \frac{1}{4\pi} \int d^3x\, d^3x' \frac{[\nabla \times \mathbf{A}(x)] \cdot [\nabla \times \mathbf{A}(x')]}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{4\pi} \int d^3x\, d^3x' \frac{[\nabla \cdot \mathbf{A}(x)][\nabla \cdot \mathbf{A}(x')]}{|\mathbf{x} - \mathbf{x}'|} + \text{surface terms.} \tag{6}$$

Each of the two terms is positive; hence (up to the surface term question) we can minimize the integral of $\mathbf{A}^2$ by choosing $\nabla \cdot \mathbf{A} = 0$. With this choice the integral of $\mathbf{A}^2$ is minimal in accord with our above remarks and is expressed only in terms of the magnetic field $\nabla \times \mathbf{A}$:

$$A^2_{\min} = \frac{1}{4\pi} \int d^3x\, d^3x' \frac{\mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \text{surface terms.} \tag{7}$$

Thus we can trade, so to speak, apparent locality for explicit gauge invariance.

It will be seen that the arguments of this section carry over to four (or more) dimensions directly, as long as the metric is Euclidean. For example, in four dimensions Eq. (6) becomes

$$\int A^2(x)\, d^4x = \frac{1}{2\pi^2} \int d^4x\, d^4x' \frac{[F_{\mu\nu}(x)][F_{\mu\nu}(x')]}{(x - x')^2} + \frac{1}{2\pi^2} \int d^4x\, d^4x' \frac{[\partial_\mu A_\mu(x)][\partial_\nu A_\nu(x')]}{(x - x')^2} + \text{surface terms.} \tag{8}$$

Again setting the second term to zero, we obtain the four dimensional analog of Eq. (7).

Quantum field theory.—Returning now to quantum field theory and vacuum condensates, we examine the suggestion that $A^2_{\min}$, now the expectation value of an operator, is sensitive to or measures the presence of topological structures in some way.

A simple model we can investigate in this regard is "photodynamics," i.e., the theory with the Lagrangian density

$$L = \frac{1}{4e^2} (F_{\mu\nu})^2. \tag{9}$$

This model can be studied in two realizations, compact and noncompact. While the noncompact realization is just the theory of free photons, it is known that the compact realization has nontrivial properties, including a phase transition near $e^2 \approx 1$ with a condensation of magnetic monopoles [5] (for review, see, e.g., [6]). Since the monopoles are the sources of nonzero magnetic flux, we would expect $A^2_{\min}$ to be sensitive to the phase transition.

We can test these ideas in a numerical simulation by considering the difference of $A^2_{\min}$ calculated in the two realizations. We take the noncompact theory, given by the action (we work in four Euclidean dimensions)

$$S_{\mathrm{non}}(F) = \frac{1}{4e^2} \int d^4x\, (F_{\mu\nu})^2, \tag{10}$$

and the compact theory where

$$S_{\mathrm{com}}(F) = \frac{1}{2e^2} \int d^4x\, [1 - \cos(F_{\mu\nu})^2], \tag{11}$$

and we examine the difference

$$\zeta(e^2) = \int \mathcal{D}A\, A^2 e^{-S_{\mathrm{com}}} - \int \mathcal{D}A\, A^2 e^{-S_{\mathrm{non}}}, \tag{12}$$

where $\int A^2$ is at its minimum for each gauge equivalent configuration.

We do this in a lattice formulation, using a 12^4 lattice with periodic boundary conditions. A^2 is then measured in units of the lattice spacing. $\mathcal{D}A$ is normalized so that $\int \mathcal{D}A\, e^{-S} = 1$, with $A(x)$ running essentially from $-\infty$ to $+\infty$ in the noncompact case and from $-\pi$ to $+\pi$ in the compact case. The minimum A^2 condition is enforced by an iterative procedure: given a certain A configuration on the lattice links, a gauge function $\alpha(x)$ (giving a new potential, $A - \nabla\alpha$) is repeatedly adjusted so as to reduce the volume integral of A^2. Each pass works outward from

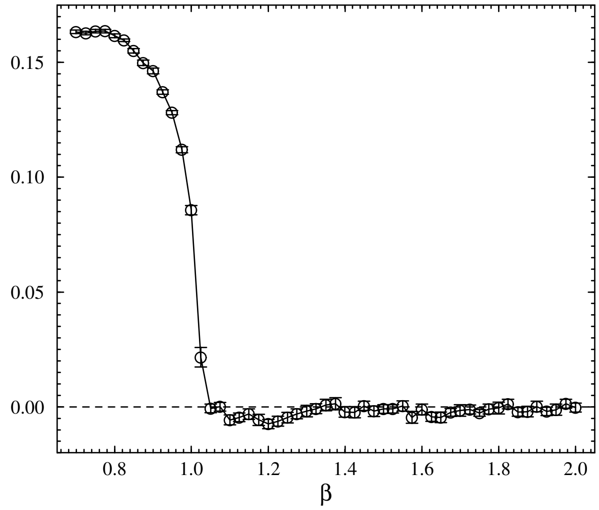

FIG. 1. $\zeta(e^2)$ in units of the lattice spacing as a function of $\beta = 1/e^2$ showing the phase transition at $\beta = 1/e^2 \approx 1.0$.

an arbitrary lattice point and the procedure stops when the reduction is less than a certain amount.

Figure 1 shows the results of the numerical simulation. The sharp jump in $\zeta(e^2)$ at the phase transition supports the idea that $A^2_{\min}$ is a measure of the presence of the monoples and their associated strings, present for $e^2 \geq 1$. The fact that ζ jumps to zero is related to the particularly simple aspect of this model, that the small e^2 sectors of the compact and the noncompact theory have the same behavior.

With these numerical calculations we have studied the ground state. When one inserts an external monopole, it is also possible to show the differing response of $A^2_{\min}$ in the two theories by analytic arguments [7].

Many open and interesting questions remain, particularly concerning the non-Abelian case and the role of a $d = 2$ condensate in QCD. We hope to deal with some of them in future work [7].

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[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[2] An $\langle A^2 \rangle$ condensate in QCD was considered in the past [by M. J. Lavelle and M. Schaden, Phys. Lett. B **208**, 297 (1988); M. Lavelle and M. Oleszczuk, Mod. Phys. Lett. A **7**, 3617 (1992), and references therein] in connection with gauge variant quantities like the gluon propagator, and taken to be only of relevance if color conservation is violated.

[3] Our discussion also gives us interesting insight into the history of superconductivity, where the Londons originally assumed that $\mathbf{A}$ was proportional to the current, $\mathbf{A} \sim \mathbf{J}$. This is a striking statement where apparently a gauge noninvariant quantity is put equal to a gauge invariant one. However, note $\mathbf{A} \sim \mathbf{J}$ requires (for static conditions) $\nabla\mathbf{A} = \nabla\mathbf{J} = 0$, which is the gauge condition (4). This arises in the Ginzburg-Landau or Abelian-Higgs description for superconductivity: in the free energy or Langrangian there is the term $|(\nabla + e\mathbf{A})\psi|^2$ which contains $e\mathbf{A}\mathbf{j} + e^2\mathbf{A}^2|\psi|^2$, where $\mathbf{j}$ is the ordinary but gauge noninvariant current $\psi^*\nabla\psi - \psi\nabla\psi^*$, while the full gauge invariant current is $\mathbf{J} = \mathbf{j} - 2e\mathbf{A}|\psi|^2$. If we write $\psi = \sqrt{\rho}\, e^{i\phi}$, then $\mathbf{j} \sim \rho\nabla\phi$. In the classical case of a uniform superconductor with rigid superfluid density ρ, we can write $\mathbf{j} = \nabla(\rho\phi)$. This is a pure gradient and an integration by parts puts the $\mathbf{A}\mathbf{j}$ term to zero if $\nabla\mathbf{A} = 0$. Only the $\mathbf{A}^2$ term in the Lagrangian or free energy survives and thus $\mathbf{A}$ interacts only with $\mathbf{A}\rho$. (Note, however, that $\mathbf{J}$ itself, being a local quantity, still contains $\nabla\phi$, and also that $\mathbf{A} \sim \mathbf{J}$ is meant to apply only to simple topologies.) Hence the Londons' ansatz may be viewed as the choice of the $\nabla\mathbf{A} = 0$ or minimum A^2 gauge, together with the physical input that ρ is nonzero and "rigid."

[4] If $\mathbf{A}$ is a curl-free field this relation will be recognized as the expression for the electrostatic field energy in terms of the charge density; if it is a divergence-free field, this relation will be recognized as the expression for the magnetostatic field energy in terms of the currents. The relation is essentially the momentum space identity $(\mathbf{k} \times \mathbf{A})^2 = \mathbf{k}^2A^2 - (\mathbf{k}\mathbf{A})^2$ in position space. The four dimensional version is $\frac{1}{2}(\epsilon_{\mu\nu\lambda\rho}k_\lambda A_\rho)^2 = k^2A^2 - (k \cdot A)^2$.

[5] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975).

[6] M. E. Peskin, Ann. Phys. (N.Y.) **113**, 122 (1978).

[7] F. V. Gubarev and V. I. Zakharov, hep-ph/0010096 (to be published).