Fractional Transport Equations for Lévy Stable Processes

Eric Lutz*
Max-Planck-Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany
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The influence functional method of Feynman and Vernon is used to obtain a quantum master equation for a system subjected to a Lévy stable random force. The corresponding classical transport equations for the Wigner function are then derived, both in the limits of weak and strong friction. These are fractional extensions of the Klein-Kramers and the Smoluchowski equations. It is shown that the fractional character acquired by the *position* in the Smoluchowski equation follows from the fractional character of the *momentum* in the Klein-Kramers equation. Connections among fractional transport equations recently proposed are clarified.

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In the theory of Brownian motion one is interested in the time evolution of a system coupled to a large environment. The effect of the coupling is modeled by a stochastic force $F(t)$ with a given probability density $P[F(t)]$. The dynamics of a Brownian particle of mass M in the presence of an external potential $U(x)$ is then described by the Langevin equation

$$M\ddot{x}(t) + \gamma\dot{x}(t) + U'(x) = \xi(t), \tag{1}$$

where $F(t)$ has been divided into a mean force proportional to the velocity, the friction force $-\gamma\dot{x}(t)$, plus a fluctuating part $\xi(t)$. In the usual treatment of Brownian motion [1], it is assumed that the random force is Gaussian distributed with variance $\langle\xi(t)\xi(t')\rangle = 2D\delta(t - t')$ where $D = \gamma kT$ is the diffusion coefficient, and the Langevin equation is shown to be fully equivalent to a phase-space equation—the Klein-Kramers equation. In the limit of strong friction, the inertial term in the Langevin equation can be neglected and the Klein-Kramers equation reduces to the Smoluchowski equation. However, it has become clear in recent years that many processes in nature, such as anomalous diffusion (for a review, see [2–4]), cannot be described by ordinary (Gaussian) Brownian motion. A case in point is the so-called Lévy flight with a stochastic force distributed according to Lévy stable statistics and which has been introduced in connection with superdiffusion [5,6]. Experimental observations of Lévy flights have been reported in micelle systems [7], in two-dimensional rotating flows [8], and in subrecoil laser cooling [9]. In this Letter we consider the generalization of transport equations to describe Lévy stable motion. This question has already been addressed in the past by using various methods [5,6,10–12], in particular, the continuous time random walk formalism [13,14]. However, most of these approaches were limited to coordinate space only. Here we present a derivation of the Klein-Kramers equation for a Lévy stable process. We consider both the case of a symmetric and asymmetric probability distribution. As our main tool, we employ the influence functional formalism developed by Feynman and Vernon [15,16].

If initially system and environment are not correlated then, according to Feynman and Vernon [15,16], the density operator of the system at time t can be written in coordinate representation as

$$\rho(x,x',t) = \int \mathcal{F}[x,x'] \exp\frac{i}{\hbar}\{S[x] - S[x']\} \times \rho(x_0,x_0',0)\mathcal{D}x(t)\mathcal{D}x'(t)\,dx_0\,dx_0'. \tag{2}$$

Here the entire information on the coupling to the environment is contained in the influence functional $\mathcal{F}[x,x'] = \Phi[\frac{x(t)-x'(t)}{\hbar}]$, where $\Phi[k(t)]$ is the characteristic functional of the probability density $P[F(t)]$,

$$\Phi[k(t)] = \int \exp\left\{i\int F(t)k(t)\,dt\right\}P[F(t)]\mathcal{D}F(t). \tag{3}$$

If $F(t)$ is Gaussian distributed with mean $\langle F(t)\rangle = \gamma(t)$ and variance $\langle[F(t) - \gamma(t)][F(t') - \gamma(t')]\rangle = D(t - t')$, the characteristic functional is given by

$$\Phi_G[k(t)] = \exp\left\{i\int \gamma(t)k(t)\,dt - \frac{1}{2}\iint D(t - t')k(t)k(t')\,dt\,dt'\right\}. \tag{4}$$

If we further assume that the friction force is proportional to the velocity of the system, $\gamma(t) = -\gamma[\dot{x}(t) + \dot{x}'(t)]/2$, and that the variance is delta correlated in time, $D(t - t') = 2D\delta(t - t')$, then the influence functional can be written in the form

$$\mathcal{F}[x,x'] = \exp\frac{i}{\hbar}\int\left\{-\frac{\gamma}{2}[x(t) - x'(t)][\dot{x}(t) + \dot{x}'(t)] + i\frac{D}{\hbar}[x(t) - x'(t)]^2\right\}dt. \tag{5}$$

By means of a small time expansion, Eq. (2) can be transformed into a differential equation for the density operator. Using the influence functional (5), this results in the master equation

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$$i\hbar \frac{\partial \rho(x,x',t)}{\partial t} = \left[H(x) - H(x') + \frac{\gamma}{2M}(x-x')(p_x - p_{x'}) - i\frac{D}{\hbar}(x-x')^2 \right]\rho(x,x',t), \quad (6)$$

where $H(x) = p_x^2/2M + U(x)$ is the Hamiltonian of the system. We recognize in Eq. (6) the master equation for quantum Brownian motion derived by Caldeira and Leggett using the oscillator bath model [17] (see also [18]).

In the case of a Lévy stable distribution, the characteristic functional is given by [19]

$$\Phi_L[k(t)] = \exp\left\{ i\int \gamma(t)k(t)\,dt - \int C(t)\,|k(t)|^\alpha \left[1 + i\beta\frac{k}{|k|}\tan\frac{\alpha\pi}{2}\right]dt \right\}, \quad (7)$$

where α ($0 < \alpha \leq 2$) is the characteristic exponent (or stability index) of the distribution and β ($-1 \leq \beta \leq 1$) is the asymmetry parameter [20]. We assume as before that $\gamma(t)$ is proportional to the velocity of the system and take $C(t) = D$ constant. This leads to the following quantum master equation for a Lévy stable process

$$i\hbar \frac{\partial \rho(x,x',t)}{\partial t} = \left[H(x) - H(x') + \frac{\gamma}{2M}(x-x')(p_x - p_{x'}) - i\frac{D}{\hbar^{\alpha-1}}|x-x'|^{\alpha-1}\left(|x-x'| + i\beta\tan\frac{\alpha\pi}{2}(x-x')\right) \right]\rho(x,x',t). \quad (8)$$

For $\alpha = 2$ the master equation (8) reduces to the Caldeira-Leggett equation (6). In order to obtain the corresponding classical transport equation, we introduce the Wigner transform of the density matrix

$$f(q,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dr\,\exp\left[-\frac{ipr}{\hbar}\right] \times \rho\left(q + \frac{r}{2}, q - \frac{r}{2}, t\right). \quad (9)$$

Applying the Wigner transform to Eq. (8) and keeping only terms in leading order in $\hbar$, we obtain the equation

$$\frac{\partial f}{\partial t} = -\frac{p}{M}\frac{\partial f}{\partial q} + U'(q)\frac{\partial f}{\partial p} + \frac{\gamma}{M}\frac{\partial}{\partial p}(pf) + \gamma kT\left[\frac{\partial^\alpha f}{\partial |p|^\alpha} + \beta\tan\frac{\alpha\pi}{2}\frac{\partial}{\partial p}\frac{\partial^{\alpha-1} f}{\partial |p|^{\alpha-1}}\right], \quad (10)$$

where we have introduced the Riesz fractional derivative which is defined through its Fourier transform as [21,22]

$$-\frac{\partial^\alpha}{\partial |p|^\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\exp[-ipy]\,|y|^\alpha. \quad (11)$$

The fractional equation (10) for the distribution function $f(q,p,t)$ describes the complete dynamics of the Brownian system in phase space, for both symmetric and asymmetric Lévy stochastic forces. In the latter case, we observe an additional contribution to the friction term which may be of relevance for the description of anomalous transport in anisotropic media [23]. For $\alpha = 2$ we recover the ordinary Klein-Kramers equation. For the particular case of a symmetric distribution, an equation similar to Eq. (10), expressed with a variant of the Weyl derivative, has been obtained by Peseckis [11]. It is worthwhile to note that the fractional character in Eq. (10) is carried by the *momentum*. In the limit of high friction, one may exploit the rapid relaxation of the momentum distribution to a stationary distribution, to write down a simplified equation for the reduced distribution function in configuration space, $\hat{f}(q,t) = \int dp\, f(q,p,t)$. Employing the systematic expansion method developed in Ref. [24], we obtain the following fractional extension of the Smoluchowski equation [the derivation of Eq. (12) will be sketched below]

$$\frac{\partial \hat{f}}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial q}[U'(q)\hat{f}] + \frac{kT}{\gamma}\left[\frac{\partial^\alpha \hat{f}}{\partial |q|^\alpha} + \beta\tan\frac{\alpha\pi}{2}\frac{\partial}{\partial q}\frac{\partial^{\alpha-1}\hat{f}}{\partial |q|^{\alpha-1}}\right]. \quad (12)$$

We see that the fractional character has been transferred to the *position*. This observation settles an apparent point of confusion in the literature. Indeed, in most of the phenomenological fractional generalizations of transport equations to phase space, it is unclear whether the fractional derivative should be taken with respect to position or momentum or even both (see, e.g., the discussion in Refs. [25,26]). Our findings show that for a Lévy flight the fractional character acquired by the position in the Smoluchowski equation follows from the fractional character of the momentum in the Klein-Kramers equation. An important consequence of Eq. (10) is that, for a harmonic potential, it does not possess a stationary solution for $\alpha \neq 2$ (contrary to the claim made in Ref. [11]): Since the variance of the Lévy force is divergent, an infinite amount of energy which cannot be balanced by the dissipation is supplied to the system [10]. An approximate stationary solution can be found only in the limit of very large friction. It is given by the product of the stationary momentum distribution with the stationary position solution of Eq. (12). Both are Lévy distributions, which means that their variances and all higher moments are infinite (they are finite in the Gaussian case $\alpha = 2$).

Let us now discuss the connections of the fractional Smoluchowski equation (12) to the equations considered in the literature. We first begin with the case of a symmetric Lévy force. Equation (12) with $\beta = 0$ has been studied in detail in Ref. [6] for the cases of a free flight, a particle subjected to a constant force and to a linear Hookean force. Moreover, it is interesting to note that Eq. (12) has been recently derived in Ref. [14] from a generalized master equation for a nonhomogeneous random walk. The corresponding fractional diffusion equation obtained by setting $U(q) = 0$ has also been considered in Ref. [22]. On the other hand, for an asymmetric random force, $\beta \neq 0$, the Smoluchowski equation (12) is a generalization to a velocity dependent damping force of a fractional diffusion equation recently obtained in Ref. [12] starting from a Langevin-like equation.

In a recent paper, Kusnezov *et al.* [25] proposed a fractional Klein-Kramers equation which was obtained as the classical limit of what they call a quantum Lévy process. Their derivation is based on a microscopic random-matrix model for a system coupled to a chaotic environment. These authors showed that for an environment with constant average level density [or equivalently with infinite temperature $\beta_T = (kT)^{-1} = 0$], the reduced density matrix displays the behavior of a free Lévy flight. However, the dynamics described by their fractional transport equation for finite temperature $\beta_T \neq 0$ is unknown. Let us now examine that point. The characteristic functional corresponding to their quantum master equation can be written as

$$\Phi_{QL}[k(t)] = \exp\left\{ i \int \gamma(t) \frac{\alpha}{2} \operatorname{sgn}k(t) \, |k(t)|^{\alpha-1} dt - \int C(t) \, |k(t)|^{\alpha} \, dt \right\}. \tag{13}$$

By comparing expression (13) to the characteristic functional of a symmetric Lévy flight (7), we observe (i) that the second terms on the right-hand side, those describing the fluctuation of the stochastic force, are equal, but (ii) that the first terms, which are related to the mean, are different. Since the latter are responsible for the dissipation, this implies that two expressions are identical only for vanishing friction. Note that this is in agreement with the results of Ref. [25], since the limit of vanishing friction precisely corresponds to $\beta_T = 0$, as can be easily seen from their Eq. (28). The fractional Kramers equation (27) (with $\beta_T \neq 0$) given in Ref. [25] thus describes a system subjected to a symmetric Lévy force, but with a mean friction force that is *different* from that of a Lévy flight. It is straightforward to determine the mean value $\langle F_{QL}(t)\rangle$ for the process defined by Eq. (13). It is given at a particular time $t = t_0$ by [16]

$$\langle F_{QL}(t_0)\rangle = -i \left. \frac{\delta \Phi_{QL}[k(t)]}{\delta k(t_0)} \right|_{k=0}. \tag{14}$$

Since $\alpha - 1 \leq 1$, the mean force $\langle F_{QL}(t_0)\rangle$ is divergent [it is finite and equal to $\gamma(t_0)$ only for $\alpha = 2$]. Hence both the first and the second moment of the process investigated by Kusnezov *et al.* are divergent. This has to be contrasted with the normal Lévy flight, where the mean is finite and equal to $\langle F_L(t_0)\rangle = \gamma(t_0)$ for all values of α. In addition, the quantum Lévy process has a stationary solution for a harmonically bound particle (since the damping force is infinite, such a solution is expected to exist). It is of the form $\exp[-\alpha(p^2/M + M\omega^2 q^2)/4kT]$. Note that this distribution is *Gaussian*. For $\alpha = 2$ it reduces to the Boltzmann distribution.

Finally, let us mention that fractional Klein-Kramers equations, based on Riemann-Liouville fractional calculus, have been recently proposed in Refs. [26,27]. These equations lead to nonexponential damping which can be expressed in terms of Mittag-Leffler functions. Their steady state solution is given by the Boltzmann distribution.

We now return to the derivation of Eq. (12). For simplicity we will consider only the case of a symmetric probability distribution. For large γ, the dynamics of the Klein-Kramers equation (10) is dominated by the term which contains the operator

$$C = \frac{\partial}{\partial p} p + MkT \frac{\partial^{\alpha}}{\partial |p|^{\alpha}}. \tag{15}$$

We shall look for an approximation to leading order in $(\gamma/M)^{-1}$ of Eq. (10) by using an eigenvalue method [24]. We denote by $\varphi_n(p)$ the eigenfunctions of the operator C and by $-n(n = 0, 1, 2, \ldots)$ the corresponding eigenvalues. We define the following raising and lowering operators $a_+ = -kT\partial/\partial p$ and $a_- = MD_p^{\alpha} + p/kT$, where the operator D_p^{α} obeys $\partial D_p^{\alpha}/\partial p = \partial^{\alpha}/\partial|p|^{\alpha}$. The two operators a_+ and a_- satisfy $C = -a_+ a_-$ and $[a_-, a_+] = 1$ and we have further the ladder relations $a_+\varphi_n(p) = (n+1)\varphi_{n+1}(p)$ and $a_-\varphi_n(p) = (1 - \delta_{n,0})\varphi_{n-1}(p)$. Next we look for a solution of Eq. (10) in the form

$$f(q,p,t) = \hat{f}(q,t)\varphi_0(p) + \left[\frac{\gamma}{M}\right]^{-1} f^{(1)}(q,p,t) + \left[\frac{\gamma}{M}\right]^{-2} f^{(2)}(q,p,t) + \cdots \tag{16}$$

and

$$\frac{\partial \hat{f}(q,t)}{\partial t} = \left(\partial^{(0)} + \left[\frac{\gamma}{M}\right]^{-1} \partial^{(1)} + \left[\frac{\gamma}{M}\right]^{-2} \partial^{(2)} + \cdots\right) \hat{f}(q,t), \tag{17}$$

where the $\partial^{(i)}$ are linear differential operators which are determined as follows: we substitute Eqs. (16) and (17) into Eq. (10) and separate the different orders in $(\gamma/M)^{-1}$. The integrability condition then yields for the two lowest orders

$$\partial^{(0)} = 0 \quad \text{and} \quad \partial^{(1)} = \frac{kT}{M} \frac{\partial}{\partial q} \left[D_q^{\alpha} + \frac{1}{kT} U'(q) \right]. \tag{18}$$

The operator $\partial^{(1)}$ is precisely the one appearing in the Smoluchowski equation (12).

To conclude, using influence functional methods, we gave the first derivation of a quantum master equation for symmetric and asymmetric Lévy flights with viscous damping. By taking the classical limit, we then obtained an extension of the Klein-Kramers equation containing fractional derivatives with respect to *momentum.* In the limit of strong damping, this equation was shown to reduce to a fractional Smoluchowski equation with fractional derivatives with respect to *position*. Furthermore, for symmetric Lévy stable laws, our results are in agreement with those of Kusnezov *et al.* in the limit of vanishingly small friction. For nonzero friction, we found that the process described by their fractional Kramers equation possesses a divergent mean damping force. In the opposite limit of strong friction, we recovered the fractional Fokker-Planck equations considered in Refs. [6,14]. Finally, for the case of asymmetric stable laws, we gave an extension to a velocity dependent friction of a fractional diffusion equation suggested in Ref. [12].

*Present address: Département de Physique Théorique, Université de Genève 24, quai Ernest Ansermet CH-1211 Genève 4, Switzerland.

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