Finite-Size Scaling for the Ising Model on the Möbius Strip and the Klein Bottle

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We study the finite-size scaling properties of the Ising model on the Möbius strip and the Klein bottle. The results are compared with those of the Ising model under different boundary conditions, that is, the free, cylindrical, and toroidal boundary conditions. The difference in the magnetization distribution function $p(m)$ for various boundary conditions is discussed in terms of the number of the percolating clusters and the cluster size. We also find interesting aspect-ratio dependence of the value of the Binder parameter at $T = T_c$ for various boundary conditions. We discuss the relation to the finite-size correction calculations for the dimer statistics.

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The systems under various boundary conditions have the same per-site free energy in the bulk limit, whereas the finite-size corrections are different. The idea of finite-size scaling (FSS) [1,2] is very important in understanding finite-size effects near the criticality. It is known that the FSS functions depend on the boundary conditions. The difference in the FSS functions for the Ising model under the periodic and free boundary conditions has been discussed in connection with the universal FSS for the percolation problem [3] and the Ising model [4]. The systems with the tilted boundary conditions were also studied [5,6]. Quite recently, Lu and Wu [7] studied dimer statistics on the Möbius strip and the Klein bottle. These two systems are other examples of interesting boundary conditions, and their topological property is unique. The dimer statistics on the Möbius strip was also studied by Brankov and Priezzhev [8]. The ground-state entropy of the Potts antiferromagnet on the Möbius strip was investigated [9]. In two dimensions the relevance of the finite-size properties to the conformal field theory is another source of interest [10–12].

In this Letter we study the FSS functions for the two-dimensional (2D) Ising model on the Möbius strip and the Klein bottle in view of increasing interest in the effect of boundary conditions for finite systems. We compare these results with those of the Ising model under different boundary conditions, that is, the free, cylindrical, and toroidal boundary conditions. A total of five boundary conditions are considered, and they are illustrated in Fig. 1. We deal with the rectangular lattice of size $L_1 \times L_2$ with the aspect ratio $a = L_1/L_2$. Thick and thin lines denote periodic and free boundaries, respectively, in Fig. 1. We impose the periodic boundary conditions in both the horizontal and vertical directions for the torus (toroidal boundary condition). The twisted periodic boundary condition is imposed in the y direction, that is,

$$f(x, y + L_2) = f(L_1 - x, y), \tag{1}$$

for the Klein bottle. The periodic boundary condition is imposed in one direction and the free one in the other direction for the cylinder (cylindrical boundary condition); the twisted periodic and the free boundary conditions are imposed for the Möbius strip. We refer to our system as a plane when we impose the free boundary conditions in both directions. The Möbius strip and the Klein bottle have a unique topological property; they have a nonorientable surface. The topological properties of five boundary conditions, the number of sides and the number of edges, are tabulated in Table I. The symmetry property under a transformation $a \to 1/a$ is also given in Table I. The system is symmetric under such a transformation for the plane and the torus.

We use the Monte Carlo simulation to study the FSS properties of the 2D Ising model with various boundary conditions near the criticality. The moments of magnetization are basic quantities for the FSS analysis. Here we focus on the Binder parameter [13], which is defined by

$$g = \frac{1}{2}\left(3 - \frac{\langle m^4\rangle}{\langle m^2\rangle^2}\right). \tag{2}$$

One may determine the critical point from the crossing point of g for different sizes as far as the corrections to FSS are negligible. The value of the Binder parameter at the critical point is not a universal quantity, and it depends on the shape of the finite systems and the boundary conditions.

We plot the temperature dependence of the Binder parameter g for several lattices with different boundary conditions in Fig. 2. The system sizes are given within the

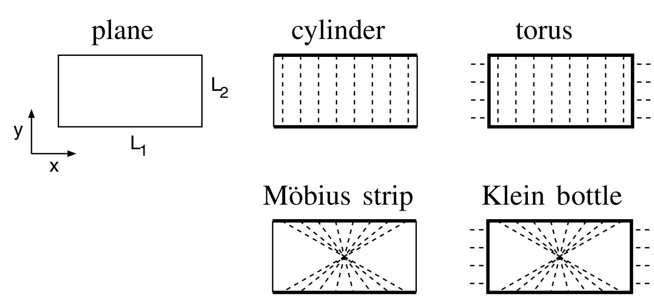

FIG. 1. Illustration of the rectangular lattice with various boundary conditions; plane, cylinder, torus, Möbius strip, and Klein bottle. Thick and thin lines denote periodic and free boundaries, respectively.

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TABLE I. The topological properties of the rectangular lattices with various geometries (boundary conditions); the number of sides and the number of edges are given. The symmetry property under the transformation $a \rightarrow 1/a$ is also given.

Geometry	No. of sides	No. of edges	$a \rightarrow 1/a$
Klein bottle	1	0	no
Möbius strip	1	1	no
Torus	2	0	yes
Cylinder	2	2	no
Plane	2	1	yes

figure, and the aspect ratio of the lattice is chosen as $a = 4$. The data of different sizes collapse on a single curve when using the scaling variable $(T - T_c)L^{1/\nu}$ for the horizontal axis. Here, $L = (L_1 \times L_2)^{1/2}$, $T_c = 2.269\ldots$, and $\nu = 1$ for the 2D Ising model; we plot the data in units of the coupling J. We obtain very good FSS behavior for g, and also find the strong dependence on the boundary conditions. The g values of the Klein bottle are larger than those of the torus. The same behavior is found in the g values of the Möbius strip and those of the cylinder.

In order to clarify the difference in the Binder parameter g, we study the magnetization distribution function $p(m)$ at $T = T_c$. We show the FSS plots of $p(m)$ for various sizes with different boundary conditions in Fig. 3. We plot $p(m)L^{-\beta/\nu}$ as a function of $mL^{\beta/\nu}$; $\beta = 1/8$ for the 2D Ising model. We again have good FSS behavior for $p(m)$ at $T = T_c$. The FSS function of $p(m)$ strongly depends on the boundary conditions. There are two sharp peaks in $p(m)$ for the Klein bottle and Möbius strip, and the secondary broad peak at about $m = 0$ becomes larger for the torus and cylinder. There is one peak about $m = 0$ for the plane.

We may understand this behavior from the number of percolating clusters and the size of clusters [14]. For the systems with large aspect ratio, the importance of the number of percolating clusters was pointed out by Hu and Lin [15]. The probability for the appearance of n percolating clusters for anisotropic lattices is of current interest [16,17]. It was revealed by Tomita *et al.* [14] that for the Ising model the combination of the percolating clusters with up spins and those with down spins gives the contribution to the broad peak at about $m = 0$ in $p(m)$. It is interesting to relate the effect of the boundary conditions to the number of percolating clusters and the size of clusters. The periodic boundary conditions may have the tendency that the size of the percolating clusters becomes larger compared to the free boundary conditions; the order will be reduced near a free surface. Moreover, if one twists the boundaries, more clusters mix together. To confirm this speculation, we study the percolating properties of the Ising model with different boundary conditions. Using the fact that the Ising model is mapped to the percolation problem with the bond concentration of $1 - e^{-2J/T}$ [18,19], we can assign clusters. We can then decompose the physical quantities by the number of percolating clusters [14]. In Fig. 4 we plot the fraction of lattice sites in the n percolating clusters $\langle c \rangle_n$ at $T = T_c$ for the system of size 160×40. We find from the figure that the cluster sizes become smaller for the system with free bondary conditions, and the single cluster is actually more dominant for the Klein bottle and the Möbius strip. Thus, we explain the large g values of the Möbius strip and the Klein bottle in relation to the number of percolating clusters.

The FSS functions for the Binder parameter g and the distribution function $p(m)$ depend on both the aspect ratio and the boundary condition. We plot the aspect-ratio

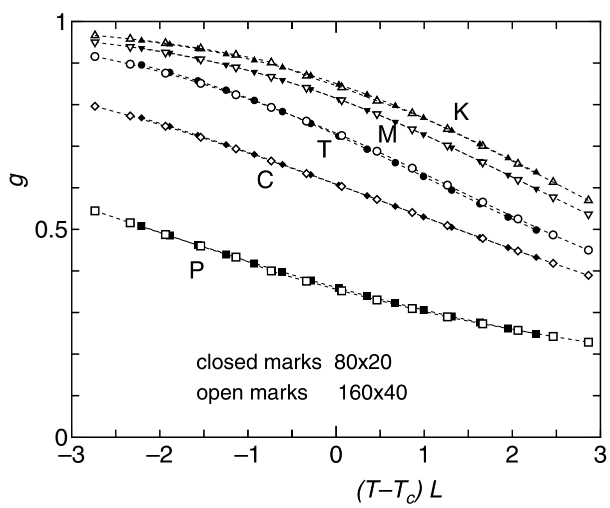

FIG. 2. Plot of g as a function of $(T - T_c)L^{1/\nu}$ for the Ising model with various boundary conditions, where $T_c = 2.269\ldots$ and $\nu = 1$. The data are plotted in units of the coupling J. Here, K, M, T, C, and P represent Klein bottle, Möbius strip, torus, cylinder, and plane, respectively. The aspect ratio a is chosen as 4.

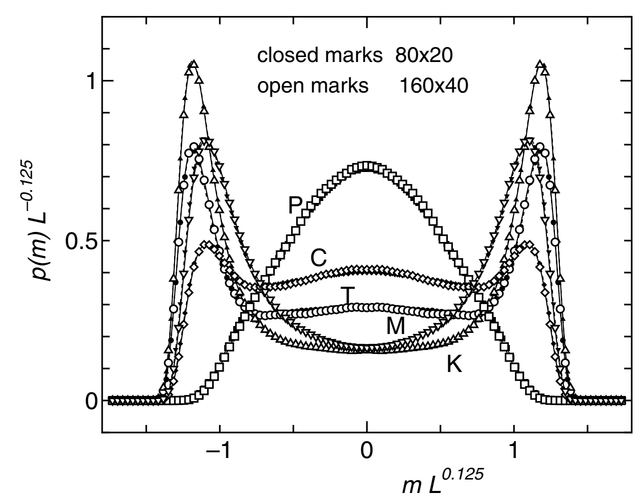

FIG. 3. Plot of $p(m)L^{-\beta/\nu}$ at $T = T_c$ as a function of $mL^{\beta/\nu}$ for the Ising model with various boundary conditions, where $\beta/\nu = 1/8$. Here, K, M, T, C, and P represent Klein bottle, Möbius strip, torus, cylinder, and plane, respectively. The aspect ratio a is chosen as 4.

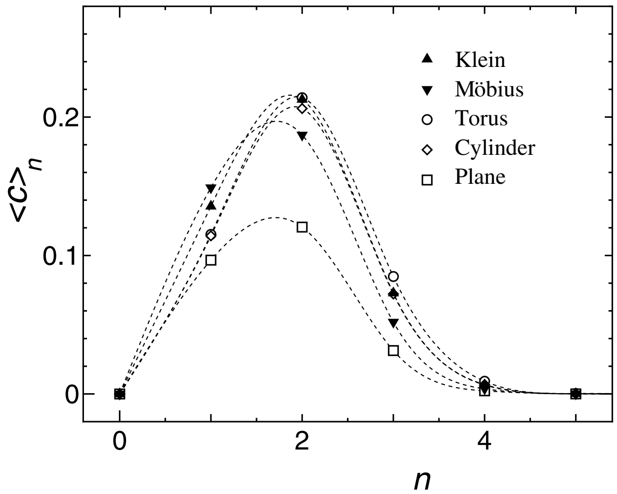

FIG. 4. Plot of $\langle c \rangle_n$ at $T = T_c$ as a function of n for the Ising model with various boundary conditions, that is, Klein bottle, Möbius strip, torus, cylinder, and plane. The system size is 160×40 ($a = 4$).

dependence of g at $T = T_c$, g_c, for various boundary conditions in Fig. 5. We use the logarithmic scales for the horizontal axis. We have several interesting observations from Fig. 5. The plot of g_c as a function of a in logarithmic scale is symmetric for the torus and the plane because of the symmetric property under the transformation $a \to 1/a$; g_c takes the maximum at $a = 1$. This is not the case for other geometries. For large enough a ($\gg 1$), the FSS properties of the Klein bottle and those of the Möbius strip become the same because the boundaries along the shorter direction determine the FSS properties of the system; for both the Möbius strip and the Klein bottle, the boundary condition along the y axis is the twisted periodic one. The FSS properties of the torus and the cylinder are the same for large enough a. In contrast, the systems with $a \leq 1$, the Klein bottle and the torus show similar FSS behavior. It is because the twisted periodic boundary or simple periodic boundary is not important for the number of percolating clusters for $a \leq 1$. It is the same situation for the Möbius strip and the cylinder for $a \leq 1$. For small enough a ($\ll 1$), the Möbius strip, the cylinder, and the plane show the same FSS properties because the boundaries along the shorter directions for these three are the same, that is, the free boundary condition.

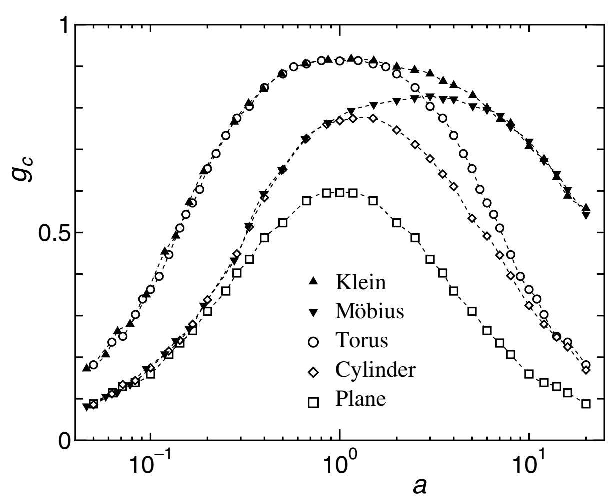

FIG. 5. Aspect-ratio dependence of the Binder parameter at $T = T_c$, g_c, for the Ising model with various boundary conditions, that is, Klein bottle, Möbius strip, torus, cylinder, and plane.

Let us compare our results with those of the dimer statistics by Lu and Wu [7], who have calculated the finite-size corrections for the dimer generating function. The finite-size correction coefficients have been tabulated in Table 1 of Ref. [7]. Their notation $M \times N$ for the system size corresponds to our $L_1 \times L_2$. In the limit of large M in their notation, which corresponds to $a \gg 1$, the finite-size correction coefficients (c_2, Δ_2) are classified into three groups, that is, Möbius strip and Klein bottle, cylinder and torus, and plane. In contrast, for large N in their notation, $a \ll 1$, the finite-size correction coefficients (c_1, Δ_1) are classified into two groups; one group is Klein bottle and torus, and the other is Möbius strip, cylinder, and plane. For both cases their results of the dimer statistics are consistent with the present results. In addition, for large $M = N$, which corresponds to $a = 1$, the correction coefficients (c_1, c_2) are relevant; then Klein bottle and torus, Möbius strip and cylinder, and plane form three groups. This result is again consistent with the present one. These observations are compatible with the conformal field theory [10–12] that the behavior of the difference in finite-size corrections for different boundary conditions is model independent.

To summarize, we have studied the FSS properties of the Ising model with various boundary conditions, that is, Klein bottle, Möbius strip, torus, cylinder, and plane. We have elucidated the difference in the magnetization distribution function $p(m)$ for various boundary conditions in terms of the number of the percolating clusters and the cluster size. We have found an interesting aspect-ratio dependence of the g value at $T = T_c$ for various boundary conditions. The FSS properties of the systems are classified into three groups for $a \gg 1$ and two groups for $a \ll 1$. For large a, the g value becomes large for the Möbius strip and the Klein bottle, which is characteristic for systems with a nonorientable surface.

There may be several directions for future study. In the dimer statistics, two identities relating dimer generating functions for Möbius strips and cylinders have been established [7]. It is desirable to explore the exact relations between the Binder parameters of the Ising systems with different boundary conditions. Anyons on the cylinder and the torus have been studied with the braid-group analysis [20]; the topology of the systems is important in the fractional quantum Hall effect. Anyons on the Möbius strip and the Klein bottle will be interesting subjects to study. The boundary condition dependence of the critical behavior of the Anderson transition was investigated [21]. It is also interesting to study the scaling functions for the Anderson transition with the Möbius and Klein-bottle boundary conditions.

Several systems on the Möbius strip and the Klein bottle have been studied in various fields of physics. Persistent currents in a Möbius ladder were studied [22], and interesting finite-size effects discussed. A general construction of correlation functions in rational conformal field theory on the Möbius strip and the Klein bottle was made in terms of three-dimensional topological quantum field theory [23]. Moreover, the matrix string theory was constructed on the Möbius strip and the Klein bottle [24]. This work on the interplay of the topology and aspect ratio in the FSS properties of the Ising model may accelerate the study of physics on a nonorientable surface.

After we submitted our paper, we found the preprint by Lu and Wu [25]. They studied the partition function of the Ising model on the Möbius strip and the Klein bottle analytically. They obtained essentially the same conclusion as that of the dimer statistics [7]. We made the detailed study of the boundary-condition dependence of the magnetization distribution function $p(m)$, which is complementary to the analytical work on the partition function.

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