Parametric Three-Wave Soliton Generated from Incoherent Light

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We show analytically and numerically that, under certain conditions, coherent localized structures can be generated and sustained from an incoherent source in quadratic nonlinear media. This phenomenon, which relies on the convection between the waves interacting in the medium, leads to the formation of a novel type of three-wave parametric soliton composed of both coherent and incoherent fields.

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Resonant three-wave interactions are ubiquitous in physics. They take place in any weakly nonlinear medium whose lowest order nonlinearity is quadratic in terms of the wave amplitudes. For this reason they are encountered in such diverse fields as plasma physics, hydrodynamics, acoustics, and nonlinear optics [1]. More recently, resonant wave mixing has been introduced to describe the so-called "superchemistry" processes in coupled atomic and molecular Bose-Einstein condensates [2]. Three-wave interactions are also relevant to the wider question of nonlinear coupled oscillator systems found in physics, chemistry, or biology [3]. Of primary importance in practice are the resonant interactions in which one of the three waves (the pump wave) is externally excited and has thus initially a much larger amplitude than the others. Two alternative theoretical approaches are usually considered to render the analysis of this situation tractable [4,5]. The phase-coherent approximation is applied when the pump wave is slowly varying, i.e., has a small spectral bandwidth $\Delta\omega$ so that $\Delta\omega\tau_0 \ll 1$, where τ_0 is the characteristic time of nonlinear interaction. In this situation the three interacting waves are assumed to be perfectly mutually coherent and their phase relationships are significant to their dynamics. On the other hand, the random phase approximation is used when the pump phase evolves rapidly (i.e., $\Delta\omega\tau_0 \gg 1$) so that its effects can be averaged over. In this case, the three waves are considered incoherent and their relative phases are not relevant to the physics of their interaction. This regime of interaction has been widely investigated, in particular, in the context of plasma physics in the framework of weak turbulence theories [4].

Coherent three-wave interactions have also been deeply investigated theoretically since the early 1970s. In particular, integrability of the governing equations was established and soliton solutions were identified [1,6]. These solitons are coherent localized structures that result from an exact balance between the energy exchanges due to the nonlinear interaction and the convection due to the group velocity differences between the waves. They have been widely investigated in the field of nonlinear optics [7–9], in particular, in the context of the self-induced transparency effect [10].

The coherent and incoherent regimes of three-wave interactions are commonly considered as being distinct. More precisely, it has recently been shown that the transition from coherent to incoherent three-wave interactions with increasing bandwidth $\Delta\omega$ is abrupt and is akin to a first-order thermodynamic phase transition [5]. In contrast with this dichotomous picture, we predict here the existence of a mixed interaction regime characterized by the coexistence of an incoherent pump wave and a coherent generated daughter wave. This prediction breaks the usual understanding of the three-wave interaction dynamics since, by virtue of the phase-sensitive nature of the interaction, an incoherent pump (with $\Delta\omega\tau_0 \gg 1$) is expected to lead naturally to incoherent daughter waves, as described by the standard random phase approximation approach. Moreover, we show that the predicted coherent-incoherent three-wave interaction process survives in the nonlinear regime of strong pump depletion in which it unexpectedly leads to the formation of a three-wave soliton composed of two incoherent waves and one fully coherent wave.

Both the process of coherent wave generation from external incoherent excitation and the associated coherent-incoherent soliton constitute the subject of the present Letter. Because nonlinear quadratic optical crystals constitute ideal test beds for the experimental verification of our predictions, we present our work in the context of nonlinear optics. However, the various physical processes described here are general and are relevant to the wide question of spontaneous organization of nonlinear ordered states in stochastic environments [11].

For concreteness, we study the three-wave interaction that couples an electromagnetic pump wave to two frequency down-converted daughter waves in a quadratic optical crystal. Assuming the spectral width of the three interacting waves to be much smaller than their respective carrier frequency ($\Delta\omega_j \ll \omega_j$, $j = 1, 2, 3$ with $\omega_3 = \omega_1 + \omega_2$) one can apply the slowly varying envelope approximation for their amplitude envelopes A_j that thus

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obey the coupled partial differential equations

$$\frac{\partial A_1}{\partial t} + v_1 \frac{\partial A_1}{\partial x} + i\beta_1 \frac{\partial^2 A_1}{\partial t^2} + \gamma_1 A_1 = \sigma_1 A_3 A_2^*, \tag{1a}$$

$$\frac{\partial A_2}{\partial t} + v_2 \frac{\partial A_2}{\partial x} + i\beta_2 \frac{\partial^2 A_2}{\partial t^2} + \gamma_2 A_2 = \sigma_2 A_3 A_1^*, \tag{1b}$$

$$\frac{\partial A_3}{\partial t} + v_3 \frac{\partial A_3}{\partial x} + i\beta_3 \frac{\partial^2 A_3}{\partial t^2} + \gamma_3 A_3 = -\sigma_3 A_2 A_1, \tag{1c}$$

where A_1, A_2, A_3 refer to the signal, idler, and pump waves, respectively. The parameters v_j, γ_j, n_j, and $k_j = n_j\omega_j/c$ are the group velocities, the damping rates, the refractive indices of the crystal, and the wave vector moduli at frequencies ω_j, respectively. The coefficients σ_j are linked to the second-order susceptibility d through the relation $\sigma_j = d v_j k_j / n_j^2$ while the dispersion coefficients are given by $\beta_j = v_j(\partial^2 k/\partial\omega^2)_j/2$. From now on we will assume that $\sigma_1 = \sigma_2 = \sigma_3/2 = \sigma$, and $\beta_j = \beta$ ($j = 1,2,3$).

A first insight into the role of convection in the coherence of the generated waves $A_{1,2}$ can be obtained by considering a dispersionless interaction ($\beta = 0$) in the linear limit of its evolution. In this limit, if we consider furthermore that pump losses are negligible, the incoherent pump wave is modeled, in the reference frame traveling at its group velocity, $z = x - v_3 t$, by a stationary single-variable stochastic function $A_3(z)$ that we shall characterize through its coherence length, λ. We can thus easily integrate Eq. (1b) along the characteristic of the idler wave and substitute the solution in Eq. (1a)

$$DA_1 = \sigma^2 \int_0^t e^{-\gamma_2(t-t')} A_3(z) A_3^*(z') A_1(x', t')\, dt', \tag{2}$$

where $D = \partial/\partial t + v_1 \partial/\partial x + \gamma_1$, $z' = z - (v_2 - v_3)(t - t')$, and $x' = x - v_2(t - t')$. The presence of the factor $A_3(z)A_3^*(z')$ in the integrand of Eq. (2) reveals the existence of a particular regime of interaction. Indeed, in the situation where the idler and pump velocities are equal, $v_2 = v_3$, one has $z' = z$, and this factor becomes $|A_3(z)|^2$, which shows that the signal evolution is independent of the pump phase fluctuations $\phi_3(z)$. This result suggests the possibility to generate, from an incoherent pump, a signal with slow phase variations, i.e., with a high degree of coherence. This phenomenon can be understood through the analysis of the idler wave, whose behavior is given by the solution of Eq. (1b): $A_2(x,t) = \sigma \int_0^t e^{-\gamma_2(t-t')} A_3(z') A_1^*(x', t')\, dt'$. When $v_2 = v_3$ one has $z' = z$ so that the function $A_3(z')$ becomes independent of time t' and can be removed from the integral, thus showing that the idler field is directly proportional to the pump field. Let us notice that this pump-idler phase-locking mechanism does not require an exact velocity matching $v_2 = v_3$. It is indeed sufficient that the velocities obey the inequality $|v_2 - v_3| \ll \lambda\gamma_2$ to be able to remove the amplitude $A_3(z')$ from the integral so that the idler follows the pump phase fluctuations.

Let us now investigate this phase-locking mechanism in the nonlinear regime of the three-wave interaction. As is well known, in the presence of a coherent pump wave the nonlinear regime is characterized by the existence of a soliton solution [6]: in the absence of dispersion and dissipation ($\beta = \gamma_j = 0$, $j = 1,2,3$), it consists of a sech-shaped envelope for the signal and idler waves and a tanh-shaped envelope for the pump wave [1,7,10]. Considering the pump-idler phase-locking mechanism, we anticipate that an incoherent pump wave can sustain the same type of solitonic structure. This prediction can be easily checked by numerical simulation of Eqs. (1). This is illustrated in Fig. 1 that shows the field envelopes after propagation over a time $t = 20\tau_0$ [$\tau_0 = 1/(\sigma e_0)$ being the characteristic time of the nonlinear interaction] when taking a pump wave of uniform modulus $|A_3| = e_0$ but with a randomly fluctuating phase. As can be verified in Fig. 1, owing to the phase-locking mechanism, the amplitude profiles $|A_j|$ are identical to those of the analytical soliton solution and the phase profile of the signal is flat. This remarkable result has been obtained for a coherence time $t_c = \lambda/v_3$ larger, smaller, or comparable to the characteristic evolution time τ_0. This soliton propagation is a good test of efficiency of the phase-locking mechanism in the nonlinear regime, but it should be noticed that it does not constitute the unique scenario of coherent/incoherent interaction. Indeed, even if our simulations show that the soliton formation is very general and occurs also with random initial conditions for both the signal and idler fields, it should be pointed out that this scenario requires an initially localized signal envelope, i.e., a limited signal pulse width. If this latter condition is not met, the signal field evolves to the coherent stationary periodic solution of the nonlinear parametric interaction.

Let us show now that an original extension of the Kolmogorov-Petrovskii-Piskunov (KPP) conjecture [12]

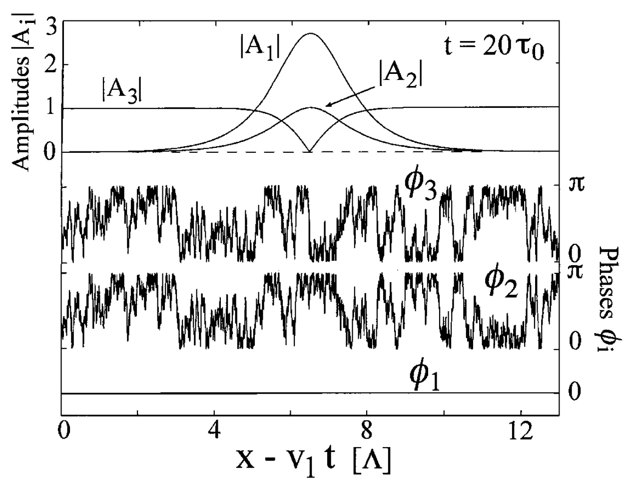

FIG. 1. Amplitudes $|A_i|$ and phases ϕ_i of the coherent/incoherent soliton. Amplitudes are given in units of $e_0 = 355$ MV/m, $\tau_0 = 0.5$ ps, $\Lambda = (v_1 - v_3)\tau_0 = 4 \times 10^{-6}$ m, $d = 20$ pm/V, $v_1 = 1.31 \times 10^8$ m/s, and $v_2 = v_3 = 1.39 \times 10^8$ m/s.

allows for the treatment of pump amplitude fluctuations. The method [8,9] consists of describing, by means of a linear analysis, the leading edge of the solitary wave as a front propagating into an unstable state. We generalize here the approach by including the effects of the stochastic driving pump field. In this view, it is more convenient to carry out the analysis in the reference frame of the pump wave defined by $(z = x - v_3 t, \tau = t)$ where the stochastic pump amplitude $A_3(z)$ is assumed to be Gaussian, ergodic, translationally invariant with zero mean $\langle A_3(z)\rangle = 0$ and the correlation function $\langle A_3(z' + z) \times A_3^*(z')\rangle = e_0^2 \exp(-|z|/\lambda)$. In the pump reference frame, the linearized Eqs. (1) read

$$\frac{\partial A_1}{\partial \tau} + V\frac{\partial A_1}{\partial z} + \gamma_1 A_1 = \sigma A_3(z) A_2^*(z,\tau), \tag{3a}$$

$$\frac{\partial A_2}{\partial \tau} + \gamma_2 A_2 = \sigma A_3(z) A_1^*(z,\tau), \tag{3b}$$

where we have neglected the dispersion effect for simplicity ($\beta = 0$). In order to take advantage of the previously discussed phase-locking mechanism, we implicitly assumed in Eqs. (3) that the pump and idler group velocities are identical ($v_3 = v_2 = v_{2,3}$). The parameter V is thus given by $V = v_1 - v_{2,3}$, which represents the amount of convection in the system. Multiplying Eq. (3a) by $[\partial/\partial\tau + \gamma_2]$, one gets a closed equation for the evolution of the signal A_1 that can easily be solved by means of the temporal Fourier expansion, which leads to

$$A_1(z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{A_1}(z = 0, \omega)\exp[f(\omega)\tau]\,d\omega, \tag{4}$$

where $f(\omega) = \{-\gamma_1 + [1 + m(z)]/[\tau_0^2(\gamma_2 - i\omega)] + i\omega\}(z/V\tau) - i\omega$ and $m(z) = (1/z)\int_0^z \epsilon(z')\,dz'$ where $\epsilon(z) = |A_3|^2(z)/e_0^2 - 1$ represents the relative intensity fluctuations of the pump field, i.e., $\langle\epsilon(z)\rangle = 0$. The function $f(\omega)$ having a saddle point located at $\omega_0 = -i\gamma_2 + i[(1 + m)z/(V\tau - z)]^{1/2}/\tau_0$, one can calculate integral (4) by the steepest descent method to get the long time behavior of the signal amplitude. For large τ we can therefore write $A_1(z,\tau) \propto \exp[f(\omega_0)\tau]$, where $f(\omega_0) = (\gamma_2 - \gamma_1)z/(V\tau) + 2[z(1 + m)(1 - z/V\tau)/(V\tau)]^{1/2}/\tau_0 - \gamma_2$.

In the spirit of the KPP approach, we now investigate the influence of the stochastic pump fluctuations on the selected signal front slope, say p^*. To this end we study the signal front in the reference frame ($\xi = z - V^*\tau$) traveling at the average velocity V^* of the solitary wave front. Since in this reference frame the front has zero mean velocity it can be described in a given interval of the variable ξ, which allows us to consider the inequality $\xi \ll V^*\tau$ in the asymptotic expression of A_1 given by the steepest descent method (i.e., for large τ). One can thus use the following expansion of the stochastic function $m(z) = m(\xi + V^*\tau) = m(V^*\tau) + (\xi/V^*\tau) \times [m(\xi) - m(V^*\tau)] + O(\xi^2/V^{*2}\tau^2)$. From the above expression of $f(\omega_0)$, we determine the behavior of the signal leading front in the new reference frame $A_1(\xi,\tau) \propto \exp(p^*\xi + \Omega\tau)$ where $\Omega = V^*(\gamma_2 - \gamma_1)/V + 2[V^*(V - V^*)]^{1/2}/(\tau_0 V) - \gamma_2$. The stationary condition for the leading front, i.e., $\Omega = 0$, yields the mean velocity of the solitary wave $V^* = V[2 + \gamma_2(\gamma_2 - \gamma_1)\tau_0^2 + 2(1 - \gamma_1\gamma_2\tau_0^2)^{1/2}]/[4 + (\gamma_2 - \gamma_1)^2\tau_0^2]$, where one can easily check that $|V^*| < |V|$.

The generalized KPP procedure describes signal fluctuations through the expression of the front slope p^*. One finds $p^* = p_{\rm coh} + \delta p$, where $p_{\rm coh} = (\gamma_2 - \gamma_1)/V + (V - 2V^*)/\{\tau_0 V[V^*(V - V^*)]^{1/2}\}$ is the deterministic slope selected in the coherent case [9], and $\delta p = m(\xi) \times [(V - V^*)/V^*]^{1/2}/(\tau_0 V)$ is the slope variation caused by pump incoherence. δp is a stochastic function that allows us to evaluate the impact of pump incoherence on the coherence of the signal field. By virtue of the ergodic and Gaussian nature of the stochastic field A_3, we can establish the following inequality $|m(\xi)| \lesssim \sqrt{\lambda/\xi}$ [13] that provides us with an upper limit for the incoherent contribution of the front slope, i.e.,

$$|\delta p| \lesssim \frac{1}{\tau_0 V}\sqrt{\frac{(V - V^*)\lambda}{V^*\xi}}\,. \tag{5}$$

Large values of $|\delta p|$ (i.e., of the order of $p_{\rm coh}$ or larger) indicate that the signal envelope is strongly influenced by the pump fluctuations and consequently turns out to be incoherent, in which case no solitonic structures can be formed. This is, in particular, the case when the parametric process takes place in the absence of any convection, i.e., when $V \simeq 0$ ($v_1 \simeq v_{2,3}$), where there is no means for the emergence of a coherent signal. Moreover, it is easy to see that as V^* increases, $|\delta p|$ decreases, which confirms the intuitive idea that the convection-induced averaging process is more efficient if the relative velocity V^* of the fluctuating pump and the solitary wave is large. Equation (5) then reveals that, as far as V and V^* are large enough to make δp negligible with respect to $p_{\rm coh}$, the signal field is coherent and a solitonic structure can be generated regardless of the pump incoherence. Note also that the role of the coherence length λ in the inequality (5) indicates that the averaging process is more efficient if the coherence length is shorter, which is in perfect agreement with the idea that, despite convection, the front slope would always follow the pump fluctuations if these were very slow (large λ).

It is interesting to check this remarkable theoretical result by means of the numerical simulation of the spontaneous generation of the parametric solitary wave in a lossy quadratic medium. We have shown in Ref. [9] that, in the presence of signal and idler losses and negligible pump loss ($\gamma_3 = 0$), a solitary wave can be spontaneously generated from a coherent pump. We investigate here this situation in the case of incoherent pumping under the condition of pump and idler velocity matching. The numerical scheme used to solve Eqs. (1) is based on the method of the characteristics [1] where the dispersion effect is finite differenced by a five-point scheme [14]. To generate numerically the stochastic amplitude

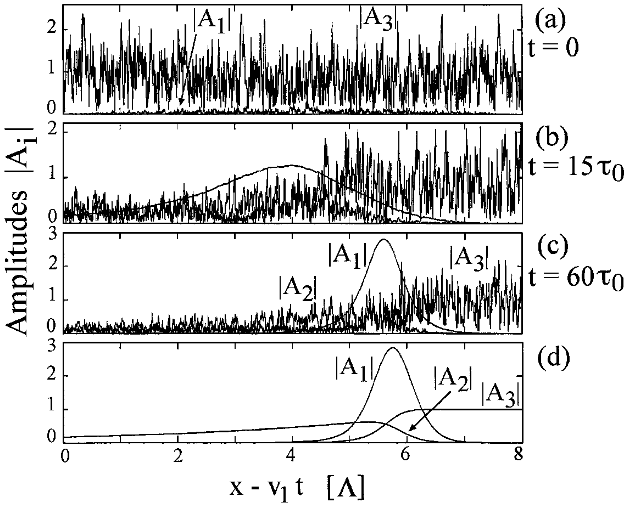

FIG. 2. Evolution of the three interacting envelopes in the reference frame traveling at the signal velocity v_1 under incoherent excitation (a)–(c); parametric solitary wave under coherent excitation (d) (the values of e_0, τ_0, and Λ are the same as in Fig. 1; the other parameters are specified in the text).

$A_3(x, t = 0)$ with the previously specified stochastic properties, we employed the Ornstein-Uhlenbeck method that is based on the solution of the Langevin equation with a δ-correlated stochastic source [11(b)]. A typical result is shown in Fig. 2 that illustrates the evolution of the initial amplitudes $A_i(x, t = 0)$ in the reference frame of the signal wave. This example would correspond to a realistic experimental situation of a *(eoo)* quasi-phase-matched $LiNbO_3$ crystal for the following central wavelengths of the three-wave packets $\lambda_1 = 2.35$ μm, $\lambda_2 = 2.9$ μm, and $\lambda_3 = 1.3$ μm. The corresponding velocities v_i are the same as those considered in Fig. 1. The injected pump intensity is $I = 24$ GW/cm^2 and the damping parameters are $\gamma_1 = 2 \times 10^{11}$ s^{-1} and $\gamma_2 = 10^{12}$ s^{-1}. The spectrum of the incident pump is very broad, $\Delta\lambda_3 \simeq 100$ nm, i.e., of the same order as that of the light bulb source recently employed to observe the spatial self-trapping of white light in photorefractive crystals [15]. Considering such fast envelope fluctuations ($t_c = \lambda/v_3 \simeq 20$ fs) we obviously have to take crystal dispersion ($k'' = 0.05$ ps^2/m) into account in the simulation of Eqs. (1).

As can be seen in Fig. 2, after a complex transient ($t > 50\tau_0$), the initially incoherent signal envelope self-structures in the form of a coherent solitary wave sustained by the incoherent pump (Figs. 2a–2c). The localized signal structure in Fig. 2c is almost identical to that obtained in the presence of a fully coherent pump (Fig. 2d) of intensity $|A_3|^2 = e_0^2$ (i.e., the mean intensity of the incoherent pump). The selected velocity V^* and front slope p^* of the solitary wave are in excellent accordance with the KPP theory. As for the conservative case of Fig. 1, the soliton formation is not the only possible scenario. In particular, if the initial random signal field is not localized, a coherent stationary signal envelope is formed.

In summary, we have shown numerically and analytically that a coherent localized structure can be generated and sustained by an incoherent pump wave in a nonlinear quadratic medium owing to convection between the interacting waves. We showed in this way the existence of a new type of coherent/incoherent parametric three-wave soliton. This new soliton exists owing to a phase-locking mechanism that occurs between the pump and one of its daughter waves provided that their group velocities are matched. Considering the simplicity of the proposed system, we can expect to observe this new phenomenon experimentally in a near future. Owing to the universality of three-wave mixing processes, our results are relevant to other branches of nonlinear science where weakly nonlinear systems are considered in stochastic environments.

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