A Convergent Series for the QED Effective Action

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The one-loop effective action of QED obtained by Heisenberg and Euler and by Schwinger has been expressed by an asymptotic perturbative series which is divergent. In this Letter we present a nonperturbative but convergent series of the effective action. With the convergent series we establish the existence of the manifest electric-magnetic duality in the one-loop effective action of QED.

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It has been well known that Maxwell's electrodynamics gets a quantum correction due to the electron loops. This quantum correction has first been studied by Heisenberg and Euler and by Schwinger a long time ago [1,2], and later by many others in detail [3,4]. The physics behind the quantum correction is also very well understood, and the various nonlinear effects arising from the quantum corrections (the pair production, the vacuum birefringence, the photon splitting, etc.) are being tested and confirmed by experiments [5,6].

Unfortunately it is also very well known that the one-loop effective action of QED has been expressed only by a perturbative series which is divergent. For example, for a uniform magnetic field B, the Euler-Heisenberg effective action is given by [2,3]

$$\Delta \mathcal{L} \simeq -\frac{2m^4}{\pi^2}\left(\frac{eB}{m^2}\right)^4 \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n+4}}{(2n+2)(2n+3)(2n+4)}\left(\frac{eB}{m^2}\right)^{2n}, \tag{1}$$

where m is the electron mass and B_n is the Bernoulli number. Clearly the series (1) is an asymptotic series which is divergent [3,4]. This is not surprising. In fact, one could argue that the effective action, as a perturbative series, can be expressed only by a divergent asymptotic series [7,8]. This suggests that only a nonperturbative series could provide a convergent expression for the effective action. There have been many attempts to improve the convergence of the series with a Borel-Pade resummation. Although these attempts have made remarkable progress for various purposes, they have not produced a convergent series so far. The purpose of this Letter is to provide a nonperturbative but convergent series of the one-loop effective action of QED. *Using a nonperturbative series expansion we prove that the one-loop effective action of QED can be expressed by*

$$\begin{aligned}\mathcal{L}_{\rm eff} = &-\frac{a^2-b^2}{2}\left(1-\frac{e^2}{12\pi^2}\ln\frac{m^2}{\mu^2}\right) - \frac{e^2}{4\pi^3}\,ab\sum_{n=1}^{\infty}\frac{1}{n} \\ &\times\left\{\coth\left(\frac{n\pi b}{a}\right)\left[\mathrm{ci}\left(\frac{n\pi m^2}{ea}\right)\cos\left(\frac{n\pi m^2}{ea}\right)+\mathrm{si}\left(\frac{n\pi m^2}{ea}\right)\sin\left(\frac{n\pi m^2}{ea}\right)\right]\right. \\ &\left.-\frac{1}{2}\coth\left(\frac{n\pi a}{b}\right)\left[\exp\left(\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(-\frac{n\pi m^2}{eb}\right)+\exp\left(-\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(\frac{n\pi m^2}{eb}-i\epsilon\right)\right]\right\},\end{aligned} \tag{2}$$

where μ is the subtraction parameter and

$$a = \frac{1}{2}\sqrt{\sqrt{F^4+(F\tilde{F})^2}+F^2},$$

$$b = \frac{1}{2}\sqrt{\sqrt{F^4+(F\tilde{F})^2}-F^2}.$$

Clearly the series is not perturbative, but convergent. The series expression has a great advantage over the divergent perturbative series. It allows us to have a massless limit. Furthermore, it has a manifest electric-magnetic duality, as we will discuss in the following.

For the scalar QED we also obtain a similar convergent series for the effective action which has a smooth massless limit and the manifest duality. Our results become important when we evaluate the effective action of QCD.

To derive the effective action let us start from the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\not{D} - m)\Psi. \tag{3}$$

With a proper gauge fixing one can show that one-loop fermion correction of the effective action is given by

$$\Delta S = i\ln\mathrm{Det}(i\not{D} - m). \tag{4}$$

So for an arbitrary constant background one has

$$\Delta\mathcal{L} = -\frac{e^2}{8\pi^2}\,ab\int_{0+i\epsilon}^{\infty+i\epsilon}\frac{dt}{t}\coth(eat)\cot(ebt)e^{-m^2t}. \tag{5}$$

Notice that the above contour of the integral is dictated by the causality. This integral expression, of course, has been known for a long time [2,3]. However, as far as we understand, the integral has been performed only in a perturbative series which is divergent (except for the special cases of a and b) [3,9].

To obtain a convergent series of the integral we need the following Sitaramachandrarao's identity [10]:

$$xy\coth x\cot y = 1 + \frac{1}{3}(x^2 - y^2) - \frac{2}{\pi}x^3y\sum_{n=1}^{\infty}\frac{1}{n}\frac{\coth(\frac{n\pi y}{x})}{(x^2 + n^2\pi^2)} + \frac{2}{\pi}xy^3\sum_{n=1}^{\infty}\frac{1}{n}\frac{\coth(\frac{n\pi x}{y})}{(y^2 - n^2\pi^2)}. \quad (6)$$

With the identity we have

$$\Delta\mathcal{L} = I_1(\varepsilon, m) + I_2(\varepsilon, m) + I_3(\varepsilon, m), \quad (7)$$

where ε is the ultraviolet cutoff parameter and [11]

$$I_1 = -\frac{1}{8\pi^2}\int_0^\infty t^{\varepsilon-3}\left(1 + e^2\frac{a^2 - b^2}{3}t^2\right)e^{-m^2t}\,dt \simeq -\frac{1}{8\pi^2}\left[\left(\frac{m^4}{2} + e^2\frac{a^2 - b^2}{3}\right)\left(\frac{1}{\varepsilon} - \gamma - \ln\frac{m^2}{\mu^2}\right) + \frac{3}{4}m^4\right],$$

$$I_2 = \frac{e^2}{4\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi b}{a})}{n}\int_0^\infty\frac{t^{\varepsilon+1}e^{-m^2t}}{t^2 + (\frac{n\pi}{ea})^2}\,dt$$

$$\simeq -\frac{e^2}{4\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi b}{a})}{n}\left[\mathrm{ci}\left(\frac{n\pi m^2}{ea}\right)\cos\left(\frac{n\pi m^2}{ea}\right) + \mathrm{si}\left(\frac{n\pi m^2}{ea}\right)\sin\left(\frac{n\pi m^2}{ea}\right)\right], \quad (8)$$

$$I_3 = -\frac{e^2}{4\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi a}{b})}{n}\int_0^\infty\frac{t^{\varepsilon+1}e^{-m^2t}}{t^2 - (\frac{n\pi}{eb})^2}\,dt$$

$$\simeq \frac{e^2}{8\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi a}{b})}{n}\left[\exp\left(\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(-\frac{n\pi m^2}{eb}\right) + \exp\left(-\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(\frac{n\pi m^2}{eb} - i\epsilon\right)\right].$$

So with the ultraviolet regularization by the modified minimal subtraction we obtain the convergent series expression (2), where we have neglected the (irrelevant) cosmological constant term.

The effective action has an imaginary part when $b \neq 0$,

$$\mathrm{Im}\mathcal{L}_{\mathrm{eff}} = \frac{e^2}{8\pi^2}ab\sum_{n=1}^{\infty}\frac{1}{n}\coth\left(\frac{n\pi a}{b}\right)\exp\left(-\frac{n\pi m^2}{eb}\right). \quad (9)$$

This is because the exponential integral $\mathrm{Ei}(-z)$ in (2) develops an imaginary part after the analytic continuation from $-z$ to z. The important point here is that the analytic continuation should be made in such a way to preserve the causality, which determines the signature of the imaginary part in (9). The physical meaning of the imaginary part is well known [2]. The electric background generates the pair creation, with the probability per unit volume per unit time given by (9).

Clearly our series expression has a great advantage over the asymptotic series. An immediate advantage is that it naturally allows a massless limit. To see this notice that in the massless limit we have

$$I_1 \simeq -\frac{e^2}{24\pi^2}(a^2 - b^2)\left(\frac{1}{\varepsilon} - \gamma - \ln\frac{m^2}{\mu^2}\right), \qquad I_2 \simeq -\frac{e^2}{4\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi b}{a})}{n}\left[\gamma + \ln\left(\frac{n\pi\mu^2}{ea}\right) + \ln\frac{m^2}{\mu^2}\right],$$

$$I_3 \simeq \frac{e^2}{4\pi^3}ab\sum_{n=1}^{\infty}\frac{\coth(\frac{n\pi a}{b})}{n}\left[\gamma + \ln\left(\frac{n\pi\mu^2}{eb}\right) + \ln\frac{m^2}{\mu^2} + i\frac{\pi}{2}\right], \quad (10)$$

so that

$$\Delta\mathcal{L}|_{m=0} = \Delta\mathcal{L}_\infty + \Delta\mathcal{L}_{\mathrm{fin}}, \quad (11)$$

where

$$\Delta\mathcal{L}_\infty \simeq \frac{e^2}{24\pi^2}\left\{a^2 - b^2 - \frac{6ab}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\coth\left(\frac{n\pi b}{a}\right) - \coth\left(\frac{n\pi a}{b}\right)\right]\right\}\ln\frac{m^2}{\mu^2}$$

$$+ \frac{e^2}{8\pi^3}ab\sum_{n=1}^{\infty}\frac{1}{n}\left[\coth\left(\frac{n\pi b}{a}\right) + \coth\left(\frac{n\pi a}{b}\right)\right]\ln\frac{a}{b} + i\frac{e^2}{8\pi^2}ab\sum_{n=1}^{\infty}\frac{1}{n}\coth\left(\frac{n\pi a}{b}\right), \quad (12)$$

and

$$\Delta\mathcal{L}_{\rm fin} = -\frac{e^2}{8\pi^3}\,ab\sum_{n=1}^{\infty}\frac{1}{n}\left[\coth\!\left(\frac{n\pi b}{a}\right) - \coth\!\left(\frac{n\pi a}{b}\right)\right] \times\left[2\gamma + \ln\!\left(\frac{n\pi\mu^2}{ea}\right) + \ln\!\left(\frac{n\pi\mu^2}{eb}\right)\right]. \qquad (13)$$

Clearly this separation of the infrared divergence was not possible with the old asymptotic series.

An important point here is that the logarithmic infrared divergence in (12) disappears when (and only when) $ab = 0$, due to the identity

$$\frac{6}{\pi}\,ab\sum_{n=1}^{\infty}\frac{1}{n}\left[\coth\!\left(\frac{n\pi b}{a}\right) - \coth\!\left(\frac{n\pi a}{b}\right)\right] = a^2 - b^2. \qquad (14)$$

Furthermore, in this case the remaining part of (11) becomes finite (after the ultraviolet subtraction). Indeed, one finds

$$\Delta\mathcal{L}\,|_{m=0} = \begin{cases} \dfrac{e^2a^2}{24\pi^2}\left(\ln\dfrac{ea}{\mu^2} - c\right) & b = 0\,, \\ -\dfrac{e^2b^2}{24\pi^2}\left(\ln\dfrac{eb}{\mu^2} - c\right) + i\,\dfrac{e^2b^2}{48\pi} & a = 0\,, \end{cases} \qquad (15)$$

where

$$c = \gamma + \ln\pi + \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\ln n = 2.2919\ldots. \qquad (16)$$

This shows that, when $ab = 0$, the effective action of QED does not have any infrared divergence even in the massless limit. This agrees with the known result [3,4].

A remarkable feature of our effective action is that it is manifestly invariant under the dual transformation,

$$a \to -ib, \qquad b \to ia\,. \qquad (17)$$

This tells that, as a function of $z = a + ib$, the effective action is invariant under the reflection from z to $-z$. Notice that, in the Lorentz frame where $\vec{E}$ is parallel to $\vec{B}$, a becomes B and b becomes E. So the duality describes the electric-magnetic duality. To prove the duality notice that the dual transformation automatically involves the analytic continuation of the special functions ci(x), si(x), and Ei(x) in (2). With the correct analytic continuation we can establish the duality in our effective action. One might think that the duality is obvious since it immediately follows from the integral expression (5). This is not so. In fact, the integral expression is invariant under the four different transformations,

$$a \to \pm ib, \qquad b \to \pm/\mp ia\,. \qquad (18)$$

But among the four only our duality (17) survives as the true symmetry of the effective action. So the duality constitutes a nontrivial symmetry of the quantum effective action. From the physical point of view the existence of the duality in the effective action is perhaps not so surprising. But the fact that this duality is borne out from our calculation of one loop effective action is really remarkable.

One can obtain the similar results for the scalar QED. In this case the one-loop correction is given by [2,3]

$$\Delta\mathcal{L}_0 = \frac{e^2}{16\pi^2}\,ab\int_{0+i\epsilon}^{\infty+i\epsilon}\frac{dt}{t}\,\mathrm{csch}(eat)\csc(ebt)e^{-m^2t}. \qquad (19)$$

To perform the integral we introduce a new identity similar to the Sitaramachandrarao's identity (6)

$$xy\,\mathrm{csch}x\csc y = 1 - \frac{1}{6}(x^2 - y^2) - \frac{2}{\pi}x^3y\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\mathrm{csch}(\frac{n\pi y}{x})}{x^2 + n^2\pi^2} + \frac{2}{\pi}xy^3\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\mathrm{csch}(\frac{n\pi x}{y})}{y^2 - n^2\pi^2}, \qquad (20)$$

and obtain

$$\Delta\mathcal{L}_0 = J_1(\varepsilon, m) + J_2(\varepsilon, m) + J_3(\varepsilon, m)\,, \qquad (21)$$

where

$$\begin{aligned} J_1 &\simeq \frac{1}{16\pi^2}\left[\left(\frac{m^4}{2} - e^2\,\frac{a^2 - b^2}{6}\right)\left(\frac{1}{\varepsilon} - \gamma - \ln\frac{m^2}{\mu^2}\right) + \frac{3}{4}m^4\right], \\ J_2 &\simeq \frac{e^2}{8\pi^3}\,ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{csch}\!\left(\frac{n\pi b}{a}\right)\left[\mathrm{ci}\!\left(\frac{n\pi m^2}{ea}\right)\cos\!\left(\frac{n\pi m^2}{ea}\right) + \mathrm{si}\!\left(\frac{n\pi m^2}{ea}\right)\sin\!\left(\frac{n\pi m^2}{ea}\right)\right], \\ J_3 &\simeq -\frac{e^2}{16\pi^3}\,ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{csch}\!\left(\frac{n\pi a}{b}\right)\left[\exp\!\left(\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\!\left(-\frac{n\pi m^2}{eb}\right) + \exp\!\left(-\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\!\left(\frac{n\pi m^2}{eb} - i\epsilon\right)\right]. \end{aligned} \qquad (22)$$

With this we finally obtain with the modified minimal subtraction (again neglecting the cosmological term)

$$\mathcal{L}_{0\mathrm{eff}} = -\frac{a^2 - b^2}{2}\left(1 - \frac{e^2}{48\pi^2}\ln\frac{m^2}{\mu^2}\right) + \frac{e^2}{8\pi^3}ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left\{\mathrm{csch}\left(\frac{n\pi b}{a}\right)\left[\mathrm{ci}\left(\frac{n\pi m^2}{ea}\right)\cos\left(\frac{n\pi m^2}{ea}\right) + \mathrm{si}\left(\frac{n\pi m^2}{ea}\right)\sin\left(\frac{n\pi m^2}{ea}\right)\right] - \frac{1}{2}\mathrm{csch}\left(\frac{n\pi a}{b}\right)\left[\exp\left(\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(-\frac{n\pi m^2}{eb}\right) + \exp\left(-\frac{n\pi m^2}{eb}\right)\mathrm{Ei}\left(\frac{n\pi m^2}{eb} - i\epsilon\right)\right]\right\}. \tag{23}$$

The effective action develops an imaginary part,

$$\mathrm{Im}\mathcal{L}_{0\mathrm{eff}} = \frac{e^2}{16\pi^2}ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{csch}\left(\frac{n\pi a}{b}\right)\exp\left(-\frac{n\pi m^2}{eb}\right). \tag{24}$$

Observe that the effective action of the scalar QED also has the manifest duality.

The effective action of the scalar QED has a smooth massless limit. For $m \simeq 0$ we have

$$\Delta\mathcal{L}_0 \simeq \frac{e^2}{96\pi^2}\left\{a^2 - b^2 + \frac{12ab}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\mathrm{csch}\left(\frac{n\pi b}{a}\right) - \mathrm{csch}\left(\frac{n\pi a}{b}\right)\right]\right\}\ln\frac{m^2}{\mu^2} + \frac{e^2}{8\pi^3}ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left\{\mathrm{csch}\left(\frac{n\pi b}{a}\right)\left[\gamma + \ln\left(\frac{n\pi\mu^2}{ea}\right)\right] - \mathrm{csch}\left(\frac{n\pi a}{b}\right)\left[\gamma + \ln\left(\frac{n\pi\mu^2}{eb}\right)\right]\right\} - i\frac{e^2}{16\pi^2}ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{csch}\left(\frac{n\pi a}{b}\right). \tag{25}$$

But remarkably the logarithmic infrared divergence disappears completely due to the following identity:

$$\frac{12}{\pi}ab\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\mathrm{csch}\left(\frac{n\pi b}{a}\right) - \mathrm{csch}\left(\frac{n\pi a}{b}\right)\right] = b^2 - a^2. \tag{26}$$

Furthermore, the remaining part of (24) becomes finite. *This shows that our series expression of the scalar QED does not contain any infrared divergence in the massless limit, even when $ab \neq 0$.* This is really remarkable, which should be contrasted with the real QED which has a genuine infrared divergence when $ab \neq 0$.

Clearly our result should become very useful in studying the nonlinear effects of QED. More importantly our effective action provides a new method to estimate the running coupling constant nonperturbatively. This, and the comparison of our result with those of the Borel-Pade resummation, will be discussed in a separate paper [12].

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