# Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model

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We study the quantum dimer model on the triangular lattice, which is expected to describe the singlet dynamics of frustrated Heisenberg models in phases where valence bond configurations dominate their physics. We find, in contrast to the square lattice, that there is a truly short ranged resonating valence bond phase with no gapless excitations and with deconfined, gapped, spinons for a *finite* range of parameters. We also establish the presence of crystalline dimer phases.

The search for a resonating valence bond (RVB) phase of frustrated magnets, inspired by ideas of Pauling [1] and begun in earnest by Anderson [2], has been one of the recurrent themes in research on the cuprate superconductors over the past decade and somewhat more. Shortly after Anderson's 1987 paper on the cuprates [3], RVB theory bifurcated — with a gapless RVB state (with valence bonds on many length scales) pursued largely by means of gauge theoretic treatments introduced by Baskaran and Anderson [4] defining one track, while Kivelson, Rokhsar, and Sethna [5] hewed closely to the original vision and pursued the study of a short ranged RVB state. The latter proposal, of a state with exponentially decaying spin-spin correlations and no long range valence bond order, was expected to lead to a gapped collective excitation spectrum and deconfined, gapped, spinons. Unfortunately, it turned out that this was hard to arrange on the square lattice — the simplest implementation of RVB ideas, the quantum dimer model [6], exhibits crystalline order and confined spinons, except at a critical point [7]. This fact is of a piece with an instability of the paramagnetic phase of the $O(3)$ nonlinear sigma model to breaking translational symmetry due to Berry phase effects [8].

In this Letter, we report that the simplest quantum dimer model on the triangular lattice *does* possess a short ranged RVB phase with gapped collective modes, gapped deconfined spinons, and spin-charge separation in its charged excitation spectrum. We establish the presence of nearby crystalline phases with confined spinons. We also suggest that a connection, made previously by Chandra and ourselves [9], between quantum dimer models and frustrated transverse field Ising models can, in this problem, be plausibly extended to track the transition out of the RVB phase. This conjecture is closely connected to the spinon deconfinement mechanism proposed by Read and Sachdev [10,11], Wen [12], Mudry and Fradkin [13], and the ideas of Senthil and Fisher [14].

We study the Rokhsar-Kivelson quantum dimer Hamiltonian generalized to the triangular lattice (as the contrasts are instructive, we will comment on the square lattice results along the way):

$$\hat{H} = -t\hat{T} + v\hat{V} = \sum_{i=1}^{N_p} \left\{ -t \sum_{\alpha=1}^{3} (|\diagup\diagdown\rangle\langle\diagdown\diagup| + \text{H.c.}) \right.$$
$$\left. + v \sum_{\alpha=1}^{3} (|\diagup\diagdown\rangle\langle\diagup\diagdown| + |\diagdown\diagup\rangle\langle\diagdown\diagup|) \right\}. \quad (1)$$

Here the sum on $i$ runs over all of the $N_p$ plaquettes, and the sum on $\alpha$ over the three different orientations of the dimer plaquettes, namely $\diamond$ rotated by 0 and $\pm 60°$. We refer to the plaquettes with a parallel pair of dimers as flippable plaquettes. As a complete orthonormal basis set we use $\{|c\rangle \,|\, c = 1, \ldots, N_c\}$, where $|c\rangle$ stands for one of the $N_c$ possible hard-core dimer coverings of the triangular lattice. $\hat{V}$ is diagonal in this basis, with $\hat{V}|c\rangle \equiv n_{fl}(c)|c\rangle$ measuring the number, $n_{fl}(c)$, of flippable plaquettes in configuration $c$.

Rokhsar and Kivelson (RK) [6] derived the square lattice version of $\hat{H}$ as the leading effective Hamiltonian in the singlet manifold consisting of nearest neighbor valence bond coverings of the lattice by utilizing their overlaps as small parameters; subsequently, it was shown by Read and Sachdev [15] that the purely kinetic energy ($\hat{T}$) piece described the $1/N$ dynamics of the nearest-neighbor SU($N$) bipartite Heisenberg magnet in an extreme quantum limit. The validity of such approaches for Heisenberg magnets depends on whether the perturbation theories remain controlled even when their expansion parameters are $O(1)$; this will depend on details of the lattice and the Hamiltonian. The perturbative scheme of RK can be generalized to the triangular lattice; by extending the phase convention introduced by Sutherland for the square lattice [16], one can obtain $t > 0$ in Eq. (1), which we assume in the remainder.

$T \gg v, t$.—At high temperatures, but less than the gap to nonvalence bond states, static properties are obtained by a classical sum over all dimer configurations. Most crisply, consider $T = \infty$ where equal time correlators are given by unweighted averages. The square lattice problem is *critical* in this limit, with algebraically decaying dimer-dimer correlations [17], whence it is not surprising that at $T = 0$ it orders everywhere except at a point.

The important observation that motivated the present work is that the triangular lattice problem is *disordered* in this limit with exponentially decaying dimer correlations. For example, the correlation function for two parallel dimers separated by distance $x$ along a column can be computed by standard Pfaffian/Grassman methods [18] as

$$\langle n(x)n(0)\rangle = (\tfrac{1}{6})^2 + G_A^2(x) + G_B(x)G_B(-x), \tag{2}$$

$$G_{\{A;B\}}(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2}\, e^{-ik_x x}\, \frac{\{2i\sin k_x; \overline{g}(\mathbf{k})\}}{4\sin^2 k_x + |g(\mathbf{k})|^2},$$

with $g(\mathbf{k}) = i[1 - e^{-ik_x} - e^{-ik_y} - e^{-i(k_x+k_y)}]$. It is easy to see that the integrands are analytic in a finite interval about real values of the momenta, whence the two Green functions, $G_{\{A;B\}}$, decay exponentially with $x$.

This feature *already* yields a high temperature RVB phase, and makes a liquid phase far more likely at zero temperature, to which we now turn.

*Topological sectors.*—An important ingredient of the analysis of the square lattice problem is the existence of two integer winding numbers, for periodic boundary conditions, that are conserved by any local dynamics in the dimer Hilbert space [6]. In that problem it is believed that the simplest dynamics (the analog of our $\hat{T}$) is ergodic within each of the $O(L^2)$ sectors. On the triangular lattice, the situation appears to be quite different. Repeating the analysis for the square case uncovers only four sectors corresponding to different combinations of even and odd windings which will be equivalent in the thermodynamic limit. With the minimal dynamics of $\hat{T}$ we have found that large classes of configurations can be connected to each other with no further constraint. The notable exceptions are the "staggered" configuration in Fig. 1 and its symmetry related counterparts, which have no flippable plaquettes whatsoever. But even here a local four dimer rearrangement can be shown to allow a connection to other "generic" states. We conjecture therefore that a general local dimer dynamics (which may require as little as the inclusion of a four dimer move in addition to $\hat{T}$) will be ergodic in each of the four topological sectors and that the dynamics in our

model is nearly so with the exception of the dynamically disconnected nonflippable configurations. Note finally that the Perron-Frobenius theorem implies that the ground state of $\hat{H}$ in *each* sector is nodeless.

*RVB phase, $v_c < v \leq t$.*—As in the square lattice case, the triangular lattice dimer model has a RK point at $v = t$ at which the ground states are equal amplitude superpositions of all dimer coverings in a given sector. To see this, note that a lower bound for the ground state energy is obtained by considering each plaquette individually. A nonflippable plaquette is annihilated by $\hat{H}$, whereas a flippable plaquette has a potential energy of $v$ and a kinetic energy of, at best, $-t$, which implies $E_0 \geq \min\{0, N_p(v - t)\}$. The equal amplitude state $|\mathrm{RK}\rangle = N_c^{-1/2}\sum_{c=1}^{N_c} |c\rangle$ (in any sector) has energy $\langle \mathrm{RK}|H|\mathrm{RK}\rangle = (v - t)\langle n_{fl}\rangle$ which vanishes and saturates the lower bound at $v = t$. Following our statements on the sectoral organization, we conclude that with the exception of the nonflippable configurations which trivially saturate this bound, there are four topologically degenerate ground states in the thermodynamic limit. The sum over *all* ground states is, for the purposes of computing correlations diagonal in the dimer basis, equivalent to the classical dimer problem. As the staggered configurations are irrelevant to this sum, we conclude that the four generic sector states have exponentially decaying dimer correlations—i.e., they are RVB states.

We next argue that these states are representative of a phase at $T = 0$. We will argue shortly that their coexistence with the staggered states is due to a first order transition out of the RVB phase exactly at the RK point—much as in the square lattice problem. Consequently, we establish that there is a range $v_c < v \leq t$ over which the RVB character of the ground state persists. Typically, one might anticipate that a disordered ground state goes along with a gap to local excitations (as opposed to the degeneracy with globally distinct states). We have examined candidates for collective modes in the single-mode approximation and found that they are all gapped. We note that, in contrast, Rokhsar and Kivelson found that their critical RVB state supported gapless excitations they dubbed resonons [19]. These together—the disordered character of the ground state and the gap in the local excitation spectrum—are strong evidence that the RK point is part of an RVB phase which is displaced to its right but will persist to its left (in Fig. 2, inset).

To test this argument, we have carried out quantum Monte Carlo simulations (see Fig. 2) on systems up to $36 \times 36$ sites at temperatures as low as $t/30$ using the method described in Ref. [9]. As expected, we find that the dimer correlations are very short ranged and practically those of the classical dimer problem, conservatively, for $2/3 < v/t < 1$. We also find a very weak temperature dependence, suggestive of a gap. As $v$ is reduced further, the correlations begin to exhibit crystalline structure (see below). At $v = t$ we find a hysteretic (first order) transition to the staggered phase described next.
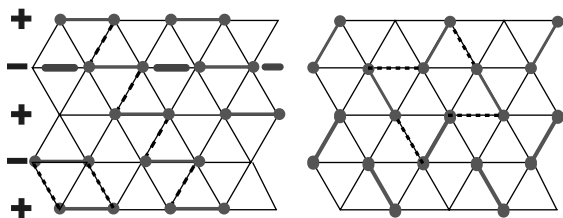


FIG. 1. Left: The columnar dimer state. The elementary dimer plaquette move generated by $\hat{T}$ is indicated in the bottom left plaquette. Such plaquette moves conserve the difference between the number of dimers in rows marked by pluses and minuses. Dimer moves $A$ and $B$ consist of shifting dimers onto the fat and dashed bonds, respectively. Right: The staggered state with the four-dimer move connecting it to other states.
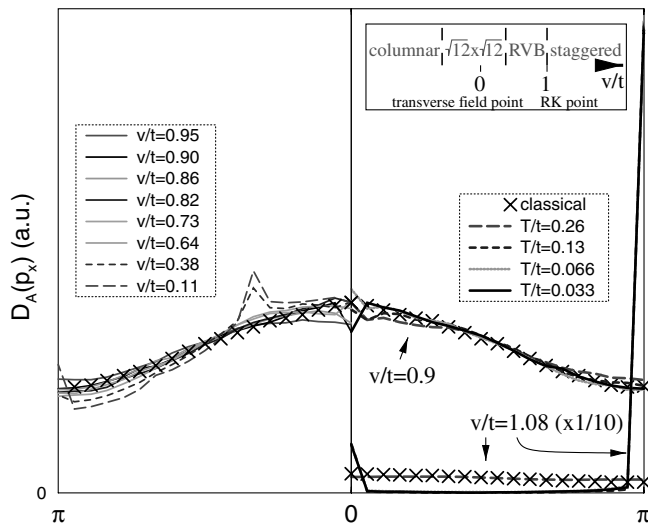
FIG. 2. Results of quantum Monte Carlo simulations of $D_A(p_x)$, the magnitude of the Fourier transform of the dimer-dimer correlation function along the $x$ axis for dimers oriented at 45° to it, for $36 \times 36$ sites. The abscissa $p_x$ runs from 0 (center) to $\pi$ on both panels. Left panel: $D_A(p_x)$ for different values of $v/t$ at $T/t = 0.033$ (arbitrary units). $D_A(p_x)$ remains close to the broad and featureless classical result (crosses) down to $v/t = 2/3$ at least (solid curves). At lower values of $v/t$ (dashed curves), short range order consistent with a $\sqrt{12} \times \sqrt{12}$ pattern arises. Right panel: temperature dependence of $D_A(p_x)$. For points in the RVB phase ($v/t = 0.9$ is shown), no dependence is discernible. At $v/t = 1.08$ (scaled down by a factor of 10 relative to $v/t = 0.9$), a Bragg peak corresponding to the staggered phase (Fig. 2) is visible at the three low temperatures (curves indistinguishable). As the temperature is raised to $T/t = 0.26$, the system enters the RVB phase. Inset: Phase diagram of the quantum dimer model on the triangular lattice.

*Staggered phase, $v > t$.*—For $v > t$ the lower bound derived previously implies $E_0 \geq 0$. As the staggered states are zero energy eigenstates of $\hat{H}$ they are the ground states in this range. This is similar to what happens in the square lattice problem, but the degeneracy is much lower in our problem [$O(L^0)$ vs $O(L)$] and the exactness of the states is here a consequence of dynamics rather than topology as mentioned earlier.

An interesting consequence of this last observation is that the excitation gap is $O(L^0)$ in system size for the staggered phase [20]. At $v/t = \infty$, the lowest energy excitations are the four dimer loop rearrangement of Fig. 1. At finite but large $v/t$, these get dressed by additional plaquette flips and acquire a dispersion. This branch of solitons is expected, by continuity, to be the relevant set of low energy excitations in the staggered phase even close to the RK point.

*Columnar phase, $-v \gg t$.*—To complete the phase diagram, we move leftwards from the RVB phase in Fig. 2, turning first to the extreme case $v/t = -\infty$. In this limit, the kinetic term $\hat{T}$ is disregarded, and the ground states are the maximally flippable states, i.e., those states $|c\rangle$ with maximal $n_{fl}(c)$.

All maximally flippable states, whose number is exponential in $L$, can be obtained by carrying out any number of operations of either type $A$ or type $B$ on the particular maximally flippable ("columnar") state as depicted in Fig. 1. Pairs of operations of either type as well as global translations and rotations can be shown to be generated by local plaquette flips, whereas a single $A$ or $B$ operation generates a state in one of the other three sectors.

Turning to the case of large but finite $-v/t$, we construct a perturbation theory using the small parameter $t/v$. Since any two maximally flippable states differ in at least $O(L)$ dimers, the degenerate perturbation theory is diagonal at any finite order. The energy shift of a state $|c\rangle$ at order $2n$ depends on the number of flippable plaquettes, $n_{fl}(c')$, of the states $|c'\rangle$ which can be reached from $|c\rangle$ by at most $n$ plaquette flips.

The result of this perturbation theory is a striking example of the phenomenon of quantum "order by disorder"—we find that it selects the columnar state (Fig. 1). States obtained by operations of types $A$ and $B$ are disfavored at 4th and 6th order, respectively.

*Transverse field Ising point, $v = 0$.*—Together with Chandra [9], we have recently shown that there exists an *exact* correspondence between the quantum dimer model at $v = 0$, and the fully frustrated transverse field Ising model (FFTFIM) on the dual hexagonal lattice at fields $\Gamma$ much smaller than the magnitude of the exchange $J$ [21]. In this limit, the quantum ground state is constructed entirely out of the ground states of the classical frustrated model and the latter are (up to Ising degeneracy) in unique correspondence with dimer coverings of its dual, triangular, lattice. In our analysis we found evidence for a low temperature crystalline "$\sqrt{12} \times \sqrt{12}$" phase exhibiting, in dimer language, a triangular superlattice with a 12 site unit cell; this phase should persist in a neighborhood of $v = 0$ (Fig. 2), but the ordering we observe numerically is not conclusive.

*Spinons.*—As noted before, the RVB phase has a gap to collective excitations. That is also true of the crystalline phases. Also of interest is the question of confinement for spinons—the gapped spin $1/2$ excitations produced by breaking a valence bond. In the dimer model, these are represented by monomers or holons that carry a spin and so *the questions of holon and spinon confinement are identical.* This question is most easily addressed by considering the free energies of states in which two monomers are held a fixed distance apart. At high temperatures, this is again a classical computation [17] and it is clear that the spinons are deconfined on the triangular lattice [22]. The contrast with the square lattice is again instructive, for there the spinons are confined at high temperatures. (However, the confinement is very weak, only logarithmic [17], which explains why the more disordered triangular lattice does not confine.) At $T = 0$ one can readily show that the state with an equal amplitude sum over dimer configurations with two spinons localized a fixed distance apart is an eigenstate at the RK point with an energy *independent*

of their separation—i.e., the spinons do not interact beyond one lattice constant [23]. By continuity, we expect spinons to be deconfined in the entire RVB phase at $T = 0$. From these considerations it also follows that *the charged excitation created by removing an electron from the system will decay into a spinon and holon.* Evidently, the holons/spinons will be confined in the crystalline phases.

*Spinon confinement transition.*—Several authors have suggested that a spinon confinement-deconfinement transition will be governed by an Ising ($Z_2$) gauge theory [10,12–14]. In our own work [9] we have found that frustrated transverse field Ising models (whose duals are precisely the Ising gauge theories) can provide a description of valence bond phases of Heisenberg antiferromagnets on their dual lattices via their connection to quantum dimer models, such as the one considered in this paper. In the current context, the geometrical identification used by us leads to an intriguing observation. As already noted, the quantum dimer model is equivalent to the $\Gamma \ll J$ limit of the FFTFIM at $v = 0$. Interestingly, the paramagnetic ground state of the FFTFIM at $\Gamma \gg J$, *when projected onto the dimer manifold,* is the equal amplitude RVB sum that is the dimer model ground state at $v = t$. This suggests the conjecture that passing between these limits in the (projected) FFTFIM gives a description of the spinon confinement and translational symmetry breaking transition that takes place at the boundary of the RVB and $\sqrt{12} \times \sqrt{12}$ phases.

In conclusion, we note that there are two previous "sightings" of a spin liquid phase on the triangular lattice in the literature, the large $N$, $Sp(N)$ analysis of Sachdev [24] and the exact diagonalization work of Misguich *et al.* [25] on an $S = 1/2$ system with a ferromagnetic two spin exchange frustrated by an antiferromagnetic four spin exchange. The latter found a fourfold disordered ground state with a spin gap, but possibly confined spinons. We hope to clarify the connection between our results and these in the near future. Finally, we note that along the lines of the analysis in [6] we expect doping to give rise, via holon condensation, to a superconducting phase on the triangular lattice.

As we were finishing this paper, there appeared Ref. [26], which reports neutron scattering evidence for deconfined spinons on an anisotropic triangular lattice. We note that the classical dimer problem in that case is also disordered, which suggests that the results of [26] could be understood along the lines of our analysis.

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[18] See, e.g., S. Samuel, J. Math. Phys. (N.Y.) **21**, 2806 (1980). Our results for the triangular lattice have not, to our knowledge, appeared previously in the literature; we expect to report more details elsewhere.
[19] Gaplessness on the square lattice relies on two ingredients in a single-mode calculation: a conservation law, which is present in our case too, for the Hamiltonian only creates and destroys pairs of parallel neighboring dimers and that the conserved quantity, defined in Fig. 2, has an extensive expectation value in a ground state. The conserved quantity is always subextensive on the triangular lattice, at most $O(L)$, and the resonons are gapped.
[20] The corresponding gap in the square lattice staggered phase is $O(L)$, which therefore has *no* local excitations in the thermodynamic limit.
[21] There are similar mappings for the square and hexagonal lattice dimer models [9].
[22] R. Moessner and S. L. Sondhi (unpublished).
[23] This is also true at the RK point on the square lattice and is implicit in the results of [6]—note the remarkable feature that raising the temperature *confines* the spinons. Elsewhere in the square lattice phase diagram, the spinons are confined.
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