Maintenance and Suppression of Chaos by Weak Harmonic Perturbations: A Unified View

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General results concerning maintenance or enhancement of chaos are presented for dissipative systems subjected to two harmonic perturbations (one chaos inducing and the other chaos enhancing). The connection with previous results on chaos suppression is also discussed in a general setting. It is demonstrated that, in general, a second harmonic perturbation can reliably play an enhancer or inhibitor role by *solely* adjusting its initial phase. Numerical results indicate that general theoretical findings concerning periodic chaos-inducing perturbations also work for *aperiodic* chaos-inducing perturbations, and in arrays of identical chaotic coupled oscillators.

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The problems of suppressing chaos by small periodic perturbations [1–6] and (to a lesser degree) preserving transient chaos by small infrequent parameter perturbations [7–9] has attracted a great deal of attention in recent years. Although such scientific areas as biology [10,11] and optics [3,12] present numerous situations where chaotic dynamics can be useful or undesirable (depending upon the specific problem considered), and in other areas, such as fluid mixing [13] and secure information processing [14], it is desirable for chaos to be generated, enhanced, and controlled during the process, the two aforementioned problems have not been considered as yet in the framework of a unified technique [15]. The aim of this Letter is to discuss a general theoretical method for the maintenance or suppression of chaos in nonautonomous systems by *solely* varying the initial phase of an applied second perturbation. Maintenance of chaos means increasing the duration of a chaotic transient or passing from transient to steady chaos.

Let us assume that a general nonautonomous system exhibits transient (steady) chaos and that we wish to maintain (intensify, i.e., to increase the leading Lyapunov exponent) or suppress the chaos by applying a (usually) small, harmonic and resonant, perturbation. The basic idea is to properly choose its amplitude (depending on the resonance order) *to drive the system to the threshold of chaos, and then to adjust its initial phase to enhance or suppress the chaos.* Analytically, the method can be discussed by considering a simple model of an unstable limit cycle affected by two weak resonant perturbations

$$x_{n+1} = [\mu + \varepsilon(f_n + \eta g_n)]x_n, \tag{1}$$

with $\mu > 1$, $\eta < 1$, $f_n = \sqrt{2}\cos n$, $g_n = \sqrt{2}\cos(n + \Psi)$, i.e., for simplicity, by choosing the main resonance case. A similar recursion relation with $\eta = 0$ is considered in Ref. [2]. Note that $\langle f_n \rangle = \langle g_n \rangle = 0$, $\langle f_n^2 \rangle = \langle g_n^2 \rangle = 1$, and $\langle f_n g_n \rangle = \cos\Psi$, angular brackets denoting the average over n. To study the effect of the two weak perturbations, one calculates the Lyapunov exponent (LE) for $\varepsilon \neq 0$: $\lambda = \mathrm{Re}\langle \mathrm{Ln}[\mu + \varepsilon(f_n + \eta g_n)]\rangle$. For small ε, the LE becomes

$$\lambda = \mathrm{Ln}\mu - \frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^2 [1 + \eta^2 + 2\eta\cos\Psi] + O(\varepsilon^3). \tag{2}$$

To clarify the effect of the second resonant perturbation, g_n, on the reduction or enhancement of instabilities (negative or positive LE, respectively), let us consider that, in the absence of the second perturbation ($\eta = 0$), we are in a *weakly* unstable initial state $\mathrm{Ln}\mu \gtrsim \frac{1}{2}(\varepsilon/\mu)^2$ such as $\lambda \simeq \lambda^+(\eta = 0) \equiv \mathrm{Ln}\mu - \frac{1}{2}(\varepsilon/\mu)^2 \gtrsim 0$. Then, *for small fixed* $\eta \neq 0$, the LE $\lambda = \lambda^+(\eta = 0) - \frac{1}{2}(\varepsilon/\mu)^2\eta(\eta + 2\cos\Psi)$ decreases when $\eta + 2\cos\Psi \geqslant 0$ and, in some range of Ψ, may become negative, thus stabilizing x—the optimal value of Ψ for stabilization being $\Psi = \Psi_{\mathrm{opt}}^{\mathrm{stab}} \equiv 0$. Contrarily, the LE increases when $\eta + 2\cos\Psi < 0$ so that the initial phase $\Psi = \Psi_{\mathrm{opt}}^{\mathrm{instab}} \equiv \pi$ yields the largest positive LE. Note that for $\eta \approx \eta_{\mathrm{threshold}} \equiv (\varepsilon/\mu)^{-1}[2\lambda^+(\eta = 0)]^{1/2}$ we obtain maximum-range intervals $[\Psi_{\mathrm{opt}}^{\mathrm{stab}} - \Delta\Psi_{\max}, \Psi_{\mathrm{opt}}^{\mathrm{stab}} + \Delta\Psi_{\max}], [\Psi_{\mathrm{opt}}^{\mathrm{instab}} - \Delta\Psi_{\max}, \Psi_{\mathrm{opt}}^{\mathrm{instab}} + \Delta\Psi_{\max}]$, of permitted initial phase for stabilization and strengthening of instabilities, respectively ($\Delta\Psi_{\max} = \pi/2$). Similarly, for $\eta > \eta_{\mathrm{threshold}}$, one sees that the corresponding ranges have shrunk, i.e., $\Delta\Psi_{\max} < \pi/2$.

To provide a rigorous formulation of the method and results, let us consider the wide and important class of dissipative systems

$$\ddot{x} + \frac{dU(x)}{dx} = -d(x,\dot{x}) + g_c(x,\dot{x})f_c(t) + g_s(x,\dot{x})f_s(t), \tag{3}$$

where $U(x)$ is a nonlinear potential, $-d(x,\dot{x})$ is a general damping force, $g_c(x,\dot{x})f_c(t)$ is the chaos-inducing perturbation, while $g_s(x,\dot{x})f_s(t)$ is an as yet undetermined suitable chaos-enhancing/suppressing perturbation, with $f_c(t), f_s(t)$ being harmonic functions with frequencies ω, Ω, and initial phases $0, \Psi$, respectively. Moreover, let us assume that the system (3) satisfies Melnikov's method (MM) requirements [16]; i.e., the time-periodic and damping terms are small amplitude perturbations of the underlying conservative system $\ddot{x} + dU(x)/dx = 0$ which has a separatrix. The application of MM to Eq. (3) gives us the *generic* Melnikov function (MF) $M_{h,h'}^{\pm}(\tau_0) = -D \pm A\,\mathrm{har}(\omega\tau_0) + B\,\mathrm{har}'(\Omega\tau_0 + \Phi_{h,h'}^{\pm})$, where $\mathrm{har}(x)$ means indistinctly $\cos(x)$ or $\sin(x)$, and D, A, B, are different non-negative functions of the corresponding parameters for each particular system. As the phase and initial time τ_0 are not fixed, we can study the simple zeros of $M_{h,h'}^{\pm}(\tau_0)$ by choosing quite freely the harmonic functions. Thus we can consider, e.g., the MF $M(t_0) = -D + A\sin(\omega t_0) - B\sin(\Omega t_0 + \Psi)$ to establish the general results. It is commonplace to note that the simple zeros of the MF give rise to transversal homoclinic points and chaotic phenomena (Smale horseshoes). The mechanism for taming chaos is then the frustration of a homoclinic (or heteroclinic) bifurcation (i.e., no horseshoe) [17], while the maintenance of chaos is achieved by moving the system from the homoclinic tangency condition *even more* than in the initial situation with no second perturbation. Let us suppose that, in the absence of any second perturbation ($B = 0$), the associated MF $M_0(t_0) = -D + A\sin(\omega t_0)$ changes sign at some t_0, i.e., $D \leqslant A$. Figure 1 shows a plot of $M_0(t_0)$. If we now let the second perturbation act on the system such that $B \leqslant A - D$, this relationship represents a sufficient condition for $M(t_0)$ to change sign at some t_0. Thus, a necessary condition for $M(t_0)$ to always have the same sign $[M(t_0) < 0]$ is $B > A - D \equiv B_{\min}$. It was previously demonstrated [6] that a sufficient condition for $B > B_{\min}$ to also be a sufficient condition for suppressing chaos is $\Omega = p\omega$ (resonance condition), $B \leqslant B_{\max} \equiv A/p^2$, p an integer, and that $M_0(t_0)$ and $-B_{\min,\max}\sin(\Omega t_0 + \Psi)$ to be in opposition (see Fig. 1). This last condition yields the optimal suppressory values of the initial phase, $\Psi_{\mathrm{opt}}^{\mathrm{sup}}$, in the sense that they allow the widest amplitude ranges for the chaos-suppressing perturbation. Now we see that imposing $M_0(t_0)$ to be *in phase* with $-B_{\min,\max}\sin(\Omega t_0 + \Psi)$ is a *sufficient* condition for $M(t_0)$ to change sign at some t_0 (see Fig. 1). This condition provides the optimal enhancer values of the initial phase, $\Psi_{\mathrm{opt}}^{\mathrm{enh}}$, in the sense that $M(t_0)$ presents its highest maximum at $\Psi_{\mathrm{opt}}^{\mathrm{enh}}$; i.e., one obtains the maximum gap from the homoclinic tangency condition. Observe that, for a given homoclinic orbit forming (part of) a separatrix, we have *in general* [i.e., for any MF $M_{h,h'}^{\pm}(\tau_0)$] that $|\Psi_{\mathrm{opt}}^{\mathrm{sup}} - \Psi_{\mathrm{opt}}^{\mathrm{enh}}| = \pi$ for each resonance order. It is straightforward to demonstrate [18] that for $B = B_{\min}$ there always exists a *maximum-range* interval $[\Psi_{\mathrm{opt}}^{\mathrm{enh}} - \Delta\Psi_{\max}, \Psi_{\mathrm{opt}}^{\mathrm{enh}} + \Delta\Psi_{\max}]$ of permitted initial phases for enhancement of chaos in the sense that, for values of Ψ belonging to that interval, the maxima of $M(t_0)$ are higher than those of $M_0(t_0)$. Similarly, for $B = B_{\max}$ there always exists a *different* maximum-range interval $[\Psi_{\mathrm{opt}}^{\mathrm{enh}} - \Delta\Psi'_{\max}, \Psi_{\mathrm{opt}}^{\mathrm{enh}} + \Delta\Psi'_{\max}]$ of allowed initial phases for maintenance of chaos, such as $\Delta\Psi'_{\max} \geqslant \Delta\Psi_{\max}$, which is a consequence of the dissipation. It must be emphasized that the definition of $\Psi_{\mathrm{opt}}^{\mathrm{enh}}$ is general; i.e., it refers to any resonance and for any MF $M_{h,h'}^{\pm}(\tau_0)$. For general separatrices, i.e., formed by several heteroclinic and (or) homoclinic loops, the above scenario holds for *each* homoclinic (heteroclinic) orbit. However, it is common to find that the different homoclinic (heteroclinic) orbits of a given separatrix yield *distinct* $\Psi_{\mathrm{opt}}^{\mathrm{enh}}$ values. This is a consequence of the

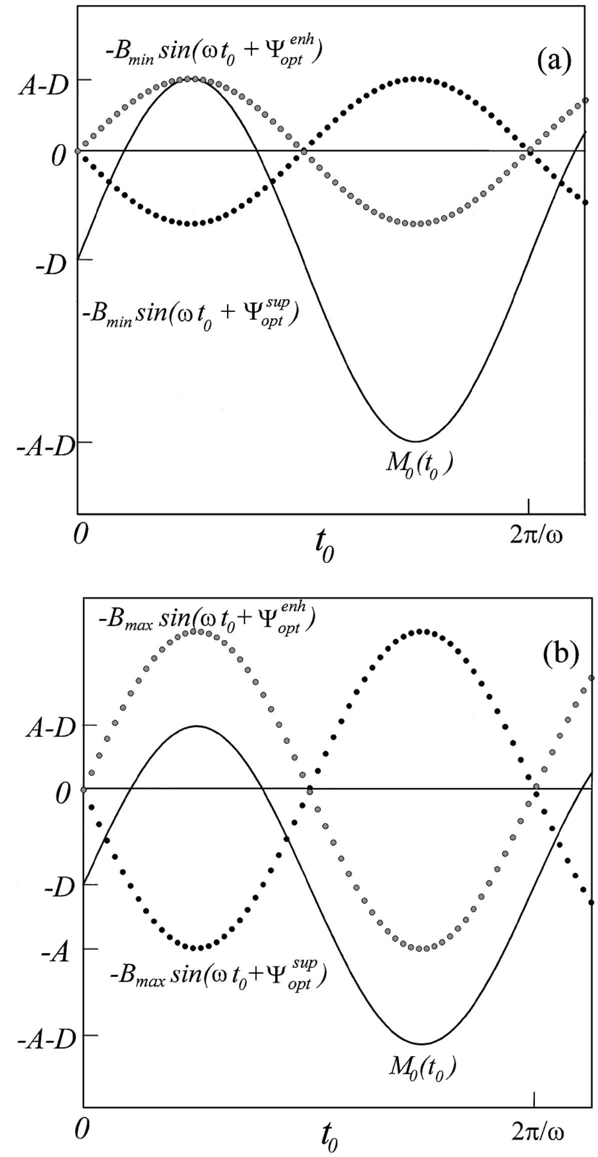

FIG. 1. Optimal suppressor and enhancer effects of a second perturbation on the initial Melnikov function $M_0(t_0) \equiv -D + A\sin(\omega t_0)$ (solid line) for the main resonance $\Omega = \omega$ case. (a) Functions $-B_{\min}\sin(\omega t_0 + \Psi_{\mathrm{opt}}^{\mathrm{sup}})$ (black dotted line) and $-B_{\min}\sin(\omega t_0 + \Psi_{\mathrm{opt}}^{\mathrm{enh}})$ (grey dotted line) vs t_0. (b) Functions $-B_{\max}\sin(\omega t_0 + \Psi_{\mathrm{opt}}^{\mathrm{sup}})$ (black dotted line) and $-B_{\max}\sin(\omega t_0 + \Psi_{\mathrm{opt}}^{\mathrm{enh}})$ (grey dotted line) vs t_0.

survival of the *symmetries* existing in the absence of the second perturbation [19]. Thus, the actual scenario is usually more complicated. For example, let $\Psi_{\text{opt},r}^{\text{sup}}, \Psi_{\text{opt},l}^{\text{sup}}$ be the optimal values associated with the right and left homoclinic orbits, respectively, of a typical separatrix with a "figure-of-eight" loop. It is straightforward to see that the best chance for enhancing chaos should now be at $\Psi_{\text{opt}}^{\text{enh}} \sim (\Psi_{\text{opt},r}^{\text{enh}} - \Psi_{\text{opt},l}^{\text{enh}})/2(\text{mod}2\pi)$. The corresponding values of $\Psi_{\text{opt}}^{\text{enh}}$ for the remaining topological types of separatrices can be readily obtained [18].

Numerical simulations of different systems show excellent agreement with theoretical predictions. Figure 2 depicts the results for a universal escape oscillator [20]

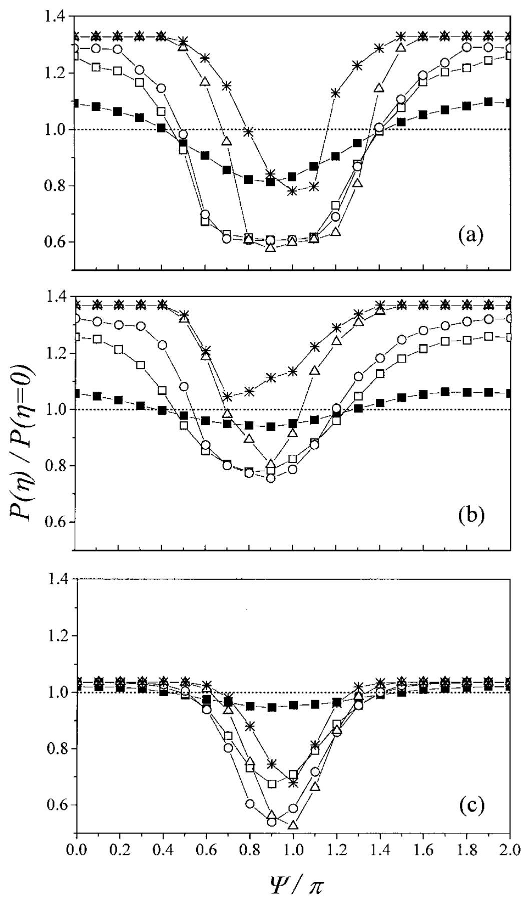

FIG. 2. Normalized escape probability $P(\eta)/P(\eta = 0)$ vs initial phase Ψ for the system $\ddot{x} + x - x^2 = -\delta\dot{x}|\dot{x}| + \eta x^2 \cos(\Omega t + \Psi) + \gamma f_c(t) + f_{\text{noise}}(t)$, where $f_c(t)$ is a chaos-inducing perturbation and $f_{\text{noise}}(t)$ is a random force, for several values of η: (■) $\eta = 0.07$, (□) $\eta = \eta_{\min} = 0.223\,323$, (○) $\eta = 0.3$, (△) $\eta = \eta_{\max} = 0.549\,748$, (*) $\eta = 0.7$. Fixed parameters: $\delta = 0.1$, $\omega = 0.85$. (a) Periodic chaos-inducing perturbation $f_c(t) = \cos(\omega t)$, $f_{\text{noise}}(t) = 0$, $\gamma = 0.05$, $\Omega = \omega$. (b) Aperiodic chaos-inducing perturbation $f_c(t) = \sin y(t)$ with $y(t)$ a chaotic response from the system $\ddot{y} + \sin y = -0.1\dot{y} + 2\cos(2\omega t)$, $f_{\text{noise}}(t) = 0$, $\Omega = \omega$. Here $\gamma = 0.1$ for having an initial escape situation similar to that of the case (a). (c) Periodic chaos-inducing perturbation in the presence of noise $f_c(t) = \cos(\omega t)$, $f_{\text{noise}}(t) = \sin\Phi(t)$ with $\Phi(t)$ a Gaussian random phase, $\gamma = 0.05$, $\Omega = \omega$.

where the separatrix is formed by a single homoclinic loop and the second perturbation is parametric. In the absence of the second perturbation ($\eta = 0$), the system presents an erosion of the safe basin (union of the bounded attractors) due to encroachment by the basin of the attractor at infinity (escaping basin) [21] for the parameters indicated in the caption to Fig. 2. Consider first the case of a harmonic chaos-inducing perturbation $f_c(t) = \cos(\omega t)$ [cf. Fig. 2(a)]. The theoretical predictions are $\eta \in]0.223\,323, 0.549\,748]$, $\Psi_{\text{opt}}^{\text{sup}} = \pi(\Psi_{\text{opt}}^{\text{enh}} = 0)$ for the inhibition (enhancement) of chaotic escape. The asymptotic behavior of the normalized escape probability, $P(\eta)/P(\eta = 0)$, as $\Psi \to \Psi_{\text{opt}}^{\text{enh}}$ indicates that the nonescaping basin has been (almost) completely destroyed. Figure 2(b) shows the corresponding results for an *aperiodic* chaos-inducing perturbation whose power spectrum exhibits a strong peak at frequency ω. Naively, one would expect that $P(\eta)/P(\eta = 0)$ is insensitive to Ψ, provided that $f_c(t)$ has no definite phase in this case. However, the existence of a sharp Fourier component, with a sufficiently high power, seems to be enough to permit the resonant second perturbation to reliably act as an inhibitor or enhancer perturbation. This represents a *new aspect of the robustness* in the maintenance and suppression of chaos by weak harmonic perturbations, which extends the known robustness against external noise—as in the case considered in Fig. 2(c). Comparison of Figs. 2(a) and 2(b) clearly indicates that the suppressory effectiveness of the second perturbations is less for an aperiodic chaos-inducing perturbation than for a periodic one, as expected. Additional numerical studies on other systems [18] indicates that the aforementioned hyper-robustness of the method discussed here is generic.

Figure 3 shows the results corresponding to a two-well Duffing oscillator, as an example of a system having a separatrix formed for several homoclinic orbits. In the absence of the second perturbation ($\eta = 0$), the system presents a strange attractor with a leading LE $\lambda^+(\eta = 0) = 0.127 \pm 0.001$ (bits/s) (dotted line in Fig. 3). The theoretical predictions, for the main resonance and the remaining parameters indicated in the caption to Fig. 3, are $\eta \in]0.094\,55, 0.4336]$, $\Psi_{\text{opt},r}^{\text{sup}} = 0$, $\Psi_{\text{opt},l}^{\text{sup}} = \pi(\Psi_{\text{opt},r}^{\text{enh}} = \pi/2, \Psi_{\text{opt},l}^{\text{enh}} = 3\pi/2)$ for the suppression (enhancement) of chaos (cf. the above discussion and Ref. [6]). The regularized response in the suppressory ranges of Ψ is invariably a period-1 solution (note the remarkable constant value of the LE). To test a certain aspect of the robustness of the method *vis-à-vis* experimental realization one can assume that the initial phase of the chaos-inducing perturbation is affected by random fluctuations, as in the case shown in Fig. 3(b). Finally, it is worth mentioning that similar results are observed [18] in one-dimensional arrays of coupled chaotic nonlinear oscillators where only the two ends are subjected to the suitable second perturbation—as in the case considered in Fig. 3(c).

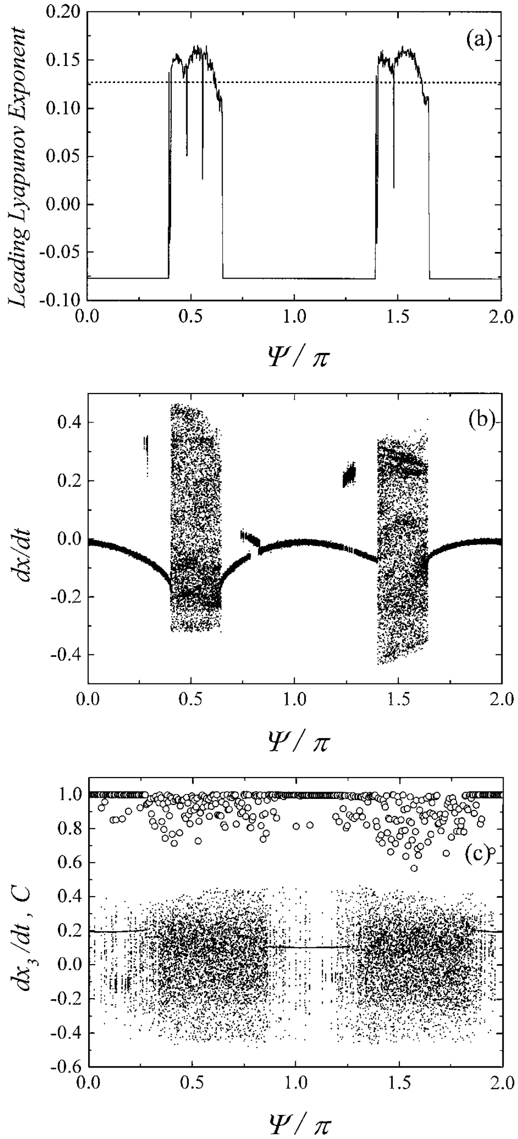

FIG. 3. (a) Leading LE vs normalized initial phase Ψ/π for the two-well Duffing oscillator $\ddot{x} - x + 4x^3 = -\delta\dot{x} + \gamma\cos(\omega t) - 4\eta x^3\cos(\Omega t + \Psi)$, $\delta = 0.154$, $\gamma = 0.095$, $\omega = \Omega = 1.1$, $\eta = (\eta_{\min} + \eta_{\max})/2 = 0.169\,525$. Dotted line represents the LE for $\eta = 0$. (b) Bifurcation diagram for the variable dx/dt vs Ψ/π for the two-well Duffing oscillator in (a) with the substitution $\omega t \rightarrow \omega t + \beta\sin\Phi(t)$, $\beta = 0.314\,15 \simeq \pi/10$, $\Phi(t)$ a Gaussian random initial phase, and the remaining parameters as in (a). (c) Bifurcation diagram for the variable dx_3/dt vs Ψ/π for the homogeneous chain of forced damped two-well Duffing oscillators governed by the equation $\ddot{x}_n - x_n + 4x_n^3 = -\delta\dot{x}_n + k(x_{n+1} - 2x_n + x_{n-1}) + \gamma\cos(\omega t)$, where $n = 1,\ldots,5$, and where only the ends are subjected to a second perturbation as in the case (a). Also plotted (○) is the correlation function $C(t) = \frac{2}{N(N-1)}\sum_{(ij)}\cos\langle x_i(t) - x_j(t)\rangle$, $N = 5$, where the summation is over all pairs of oscillators. Coupling $k = 0.1$ and the remaining parameters as in the case (a).

In summary, this theoretical and numerical study showed the initial phase of a second harmonic perturbation to play a switching role in the suppression and enhancement of chaos in nonautonomous systems. In view of the generality of this result, and the great robustness, scope, and flexibility of the technique, one can expect it to be quite readily testable by experiment. The results also confirm and extend the close relationship between the responses of a given system to aperiodic and periodic signals which have been described for the synchronization phenomenon [22]. The method discussed in this work can be directly applied to a number of important problems: control of Josephson junction arrays, mixing around a vortex in fluid mechanics, and chaotic oscillations of a satellite on an elliptic orbit are some examples.

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