Black Hole Thermodynamics from Calculations in Strongly Coupled Gauge Theory

Daniel Kabat,[1,2] Gilad Lifschytz,[3] and David A. Lowe[4]

[1]*Department of Physics, Columbia University, New York, New York 10027*
[2]*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540*
[3]*Department of Physics, Princeton University, Princeton, New Jersey 08544*
[4]*Department of Physics, Brown University, Providence, Rhode Island 02912*

(Received 3 August 2000)

We develop an approximation scheme for the quantum mechanics of N D0-branes at finite temperature in the 't Hooft large-N limit. The entropy of the quantum mechanics calculated using this approximation agrees well with the Bekenstein-Hawking entropy of a ten-dimensional nonextremal black hole with 0-brane charge. This result is in accordance with the duality conjectured by Itzhaki, Maldacena, Sonnenschein, and Yankielowicz [Phys. Rev. D **58**, 046004 (1998)]. Our approximation scheme provides a model for the density matrix which describes a black hole in the strongly coupled quantum mechanics.

DOI: 10.1103/PhysRevLett.86.1426 PACS numbers: 04.70.Dy, 11.15.Pg, 11.25.Sq

Properties of black holes in quantum theories of gravity have intrigued physicists for many years. In recent years much progress has been made in understanding some of these properties using string theory. However, because string theory is usually formulated perturbatively about a given spacetime background, it has been difficult to obtain a unified and complete description of black hole physics. In the past few years nonperturbative formulations of string theory have been proposed in terms of large-N gauge theories. These formulations give new hope for understanding some of these fundamental issues.

In particular, Maldacena's conjecture gives a promising arena to answer some of these questions. Maldacena's conjecture [1] relates string theories in anti–de Sitter backgrounds to conformal field theories in a large-N limit. However there is no "free lunch." In this framework, seemingly obvious questions on the string theory side are hard to formulate on the gauge theory side. Moreover, in the range of parameters where semiclassical string theory may be used to construct black hole geometries, the dual gauge theory is strongly coupled. This makes understanding black hole states very difficult.

The version of Maldacena duality to be considered here relates black holes in ten dimensions with 0-brane charge to supersymmetric gauged SU(N) quantum mechanics with sixteen supercharges [2]. The black holes in question have a nonzero Hawking temperature T. The dual quantum mechanics is to be taken at the same finite temperature. The black hole has a free energy, which arises from its Bekenstein-Hawking entropy,

$$\beta F = -2.52 N^2 \left(\frac{T}{(g_{\rm YM}^2 N)^{1/3}} \right)^{1.80} . \qquad (1)$$

Duality predicts that the quantum mechanics should have the same free energy. The supergravity description is expected to be valid when the curvature and the dilaton are small near the black hole horizon. This regime corresponds to the 't Hooft large-N limit of the quantum mechanics, when the dimensionless effective coupling $g_{\rm YM}^2 N/T^3$ is large.

In this Letter we describe a set of approximations that can be applied to the quantum mechanics in the regime of interest. Using these techniques we calculate the finite temperature partition function of the quantum mechanics. Over a certain range of temperature our results can be well fit by a power law,

$$\beta F \approx -0.79 - 2.0 N^2 \left(\frac{T}{(g_{\rm YM}^2 N)^{1/3}} \right)^{1.7} . \qquad (2)$$

This is in quite good agreement with the black hole prediction (1): the exponents differ by 6% while the coefficients of the power laws differ by 26%. (An additive constant appears in the approximation for the free energy. We will generally ignore this "ground state degeneracy," since it seems to be an artifact of the approximation when applied to systems with a continuous spectrum. Similar behavior was noted in [3].) We believe that this is the first nontrivial direct test of a strong/weak coupling duality that does not rely on supersymmetric nonrenormalization theorems or special properties of Bogomolnyi-Prasad-Sommerfield states. Although in this paper we are primarily interested in the thermodynamics of the quantum mechanics, our approximation scheme should also be useful for addressing questions about the spacetime structure of the black hole, perhaps along the lines of [4,5].

The basic idea is to treat the $\mathcal{O}(N^2)$ degrees of freedom of the quantum mechanics as statistically independent, using a type of mean field approximation. This assumption is motivated by the overall N^2 dependence of the free energy (1). The approximation involves constructing a trial action S_0 from the full action S. All quantities can then be systematically computed as an expansion in powers of $S - S_0$. The terms in the trial action are fixed by solving a truncated version of the Schwinger-Dyson equations

 0031-9007/01/86(8)/1426(4)$15.00

of the quantum mechanics. This procedure can be viewed as resumming an infinite number of Feynman diagrams. Since we are interested in large-N behavior, we will only resum planar diagrams. Thus, in our approximation, the overall N^2 factor in the free energy (2) and the appearance of $g_{\rm YM}^2$ only in the combination $g_{\rm YM}^2 N$ is guaranteed. The crucial test of the approximation is to obtain the correct power-law dependence of the thermodynamics on the effective dimensionless coupling $g_{\rm YM}^2 N/T^3$. Henceforth we adopt units in which $g_{\rm YM}^2 N = 1$.

We now sketch the application of the Gaussian approximation [3] to gauged SU(N) supersymmetric quantum mechanics with sixteen supercharges. Further details will appear in [6]. To avoid explicit breaking of supersymmetry by the approximation, we adopt an unconstrained superfield formulation. This ensures that we recover exact supersymmetry in the zero temperature limit. We will use an $\mathcal{N} = 2$ superspace. This means that only an SO(2) $\times$ G_2 subgroup of the SO(9) R-symmetry is manifest.

For more details on notation see [3]. $\mathcal{N} = 2$ superspace has an SO(2) R-symmetry, with spinor indices $\alpha, \beta = 1, 2$ and vector indices $i, j = 1, 2$. The $\mathcal{N} = 16$ multiplet decomposes into a set of real scalar superfields Φ_a transforming in the **7** of G_2 plus a gauge superfield Γ_α. Φ has components ϕ, ψ_α, and f, while Γ_α has components A_0, X^i, χ_α, λ_α, and d. We impose the supersymmetric gauge condition $D^\alpha \Gamma_\alpha = 0$. This sets $\partial A_0/\partial t = 0$, $d = 0$, and $\lambda = \frac{1}{2}\partial\chi/\partial t$. This is a convenient gauge fixing, as this gauge condition helps make the approximation compatible with Ward identities [6]. To the action we must add a ghost kinetic term (but no gauge fixing term).

We are interested in the finite temperature behavior of the quantum mechanics. As usual we compactify the Euclidean time coordinate on a circle of circumference β, which is identified with the inverse temperature, and expand fields in terms of Matsubara frequencies. For example, we write

$$X^i(\tau) = \frac{1}{\sqrt{\beta}} \sum_{l=-\infty}^{\infty} X_l^i e^{i2\pi l\tau/\beta}. \qquad (3)$$

Note that in Euclidean space the zero mode of the gauge field, which we denote A_{00}, survives as a physical degree of freedom (fluctuations in A_0 are eliminated by our gauge condition).

Throughout this paper we use the following expression for the free energy in this scheme:

$$\beta F \approx \beta F_0 + \langle S - S_0 \rangle_0 - \tfrac{1}{2}\langle S_{\rm III}^2 \rangle_{\rm C,0}. \qquad (4)$$

Here βF_0 is the free energy of the trial action, and $\langle\cdot\rangle_0$ denotes an expectation value computed using S_0. $S_{\rm III}$ refers to cubic terms in the original gauge plus ghost action, and the subscript C denotes a connected correlation function. It is straightforward, though tedious, to compute higher order terms in the expansion of βF. In principle, this could be used as a check on the validity of the approximation.

We make the following ansatz for the trial action:

$$S_0 = -\frac{N}{\lambda}\,{\rm Tr}(U + U^\dagger) + \sum_{l,i} \frac{1}{2\sigma_l^2}\,{\rm Tr}(X_l^i X_{-l}^i) + \sum_{l,a} \frac{1}{2\Delta_l^2}\,{\rm Tr}(\phi_l^a \phi_{-l}^a) + \cdots. \qquad (5)$$

Here all fields (except the gauge field) appear in Gaussian form. The gauge field must be treated in a special way, owing to its periodicity properties. To do this we introduced the timelike Wilson loop operator U, which can be expressed in terms of the gauge zero mode,

$$U = P e^{i\int_0^\beta A_0} = e^{i\sqrt{\beta}A_{00}}. \qquad (6)$$

This makes it manifest that, at finite temperature, $A_0 \sim A_0 + 2\pi/\beta$ is periodic. As a trial action for the gauge field we have adopted the unitary one-plaquette model action. As λ varies the trial action goes through a Gross-Witten phase transition at $\lambda = 2$ [7].

The key step in the approximation is to find a closed set of "gap" equations for the dressed propagators $\sigma_l^2, \Delta_l^2, \ldots$ appearing in (5). Again, the gauge field must be treated as a special case. All other propagators are obtained by demanding stationarity of the quantity (4). (This quantity can be identified with the two-loop 2PI effective action of [8].) Up to contributions from the gauge field, it can be shown that this procedure correctly resums all one-loop self-energy corrections to the propagators.

The gap equation for λ is obtained from the Schwinger-Dyson equation for $\langle U \rangle$ that arises from the change of variables $U \to gU$ with $g \in$ SU(N). Demanding that this equation hold with respect to the one-plaquette measure yields

$$\langle {\rm Tr}U \rangle_0 = \frac{1}{\beta^{3/2}}\,{\rm Tr}\left\langle U\left(\frac{\delta S}{\delta A_{00}} - \frac{1}{2}\frac{\delta S_{\rm III}^2}{\delta A_{00}}\right)\right\rangle_{\rm C,0}. \qquad (7)$$

This equation resums one-loop corrections to the Wilson loop, in the same sense that (4) resums one-loop corrections to the propagators. At large N the terms on the right-hand side factorize into a gauge field correlator times matter field correlators; the terms involving the gauge fields may be computed using the results of [7]. As (4) and (7) are somewhat lengthy expressions, we will not present them here.

The gap equations can be solved numerically, using the methods discussed in Appendix B of [3]. The basic strategy is to start at high temperature, where the gap equations can be solved semianalytically, then use Newton-Raphson to solve the gap equations at a sequence of successively lower temperatures.

In principle, the resulting Gaussian action contains a great deal of information about correlation functions in the quantum mechanics. But in this Letter we will just concentrate on the behavior of three basic quantities: the free energy, the Wilson loop, and the mean size of the state.

At high temperatures ($\beta \ll 1$), where the gauge theory is weakly coupled, we find that the free energy of the system is

$$\beta F = 6 \log \beta + \mathcal{O}(1). \quad (8)$$

This result can be obtained analytically: the gap equations are dominated by the bosonic zero modes, and the free energy is dominated by βF_0.

In general, for a weakly coupled theory in 0 + 1 dimensions, one would expect the free energy to behave like $\log\beta$. But note that, even though the gauge theory is weakly coupled at high temperature, the perturbation series is afflicted with IR divergences. Thus, to determine the coefficient of the logarithm (which depends on the value of the dynamically generated IR cutoff) one must resum part of the perturbation series. This is a well-known phenomenon in finite temperature field theory [9]. In any case, we expect *a priori* that the Gaussian approximation gives good results in the high temperature regime.

As the temperature is lowered the behavior of the free energy changes: at $\beta \approx 0.7$ we find that it begins to roll over and fall off as a nontrivial power of the temperature. In the range $1 < \beta < 4$ the numerical results for the free energy are well fit by (2). This fit to the numerical results is illustrated in Fig. 1. Note that supersymmetry is crucial in making such power-law behavior possible. Without supersymmetry the free energy would behave as $\beta F \approx \beta E_0$ in the low temperature regime $\beta > 1$, where E_0 is the ground state energy of the system.

We obtained (2) by performing a Levenberg-Marquardt nonlinear least-squares fit to 75 numerical calculations of the free energy, carried out in the temperature range $1 \leq \beta \leq 4$. To estimate the uncertainty in the best fit parameters we varied the window of β over which the fit was performed (fitting over the ranges $2 < \beta < 4$ and $1 < \beta < 3$), which leads to -0.79 ± 0.06, -2.0 ± 0.1, and -1.7 ± 0.2.

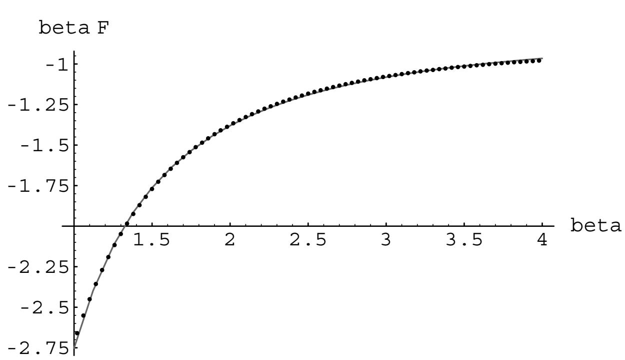

FIG. 1. The solid curve is the power-law fit (2) for βF. The data points are calculated from numerical solutions to the gap equations.

As we go to still lower temperatures, we find that the energy $\partial(\beta F)/\partial\beta$ calculated in the Gaussian approximation begins to drop below the energy of the black hole. In fact the Gaussian energy becomes negative at about $\beta = 5.8$. Ultimately, as $\beta \rightarrow \infty$, the Gaussian energy does asymptote to zero, as required by the $\mathcal{N} = 2$ supersymmetry which is manifest in the approximation. But a negative energy clearly reflects some problem with the approximation.

Fortunately, we can be rather precise about exactly where the approximation is going wrong: the difficulty is with the Schwinger-Dyson gap equation we have been using to fix the value of the one-plaquette coupling λ. Although we do not know how to write a better gap equation for λ, we can give a *prescription* for fixing λ that will allow us to obtain reasonable results at much lower values of the temperature. This may be regarded either as a check on our understanding of why the approximation is breaking down or as a way of building a model for the black hole that can be used at lower temperatures. Our prescription for fixing λ is simply that, when $\beta > 2.5$ (the midpoint of our range $1 \leq \beta \leq 4$), we choose λ so that the free energy is given by (2). The energy $E = \partial(\beta F)/\partial\beta$ calculated with this prescription is shown in Fig. 2. [One might consider other prescriptions for fixing λ. For example, the Schwinger-Dyson gap equation could be replaced with a condition of SO(9) invariance: $\langle (X^i)^2 \rangle = \langle (\phi^a)^2 \rangle$. This prescription works fine at high temperatures, but has no real solutions in the low temperature regime $\beta \gtrsim 1$. Evidently, the lack of manifest SO(9) invariance in the superfields cannot be compensated just by adjusting λ.]

With this prescription we find that λ increases monotonically with β. A Gross-Witten phase transition takes place when $\lambda = 2$; this value is reached at $\beta = 7.8$. Thus a phase transition takes place as the system moves into the supergravity regime [3].

By adopting the prescription of fitting βF to a power law, we cannot say anything about the order of the phase transition. If one takes the Schwinger-Dyson result for λ

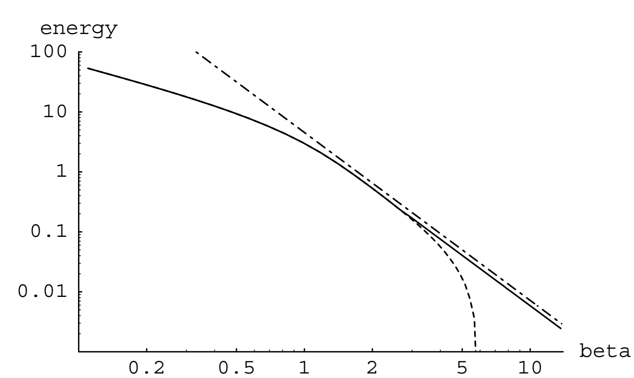

FIG. 2. Energy vs β on a log-log scale. For $\beta > 2.5$ fixing λ by fitting βF to a power law leads to the solid middle line, while the Schwinger-Dyson gap equation for lambda leads to the lower dashed line. The upper dot-dashed line is the semiclassical energy of the black hole.

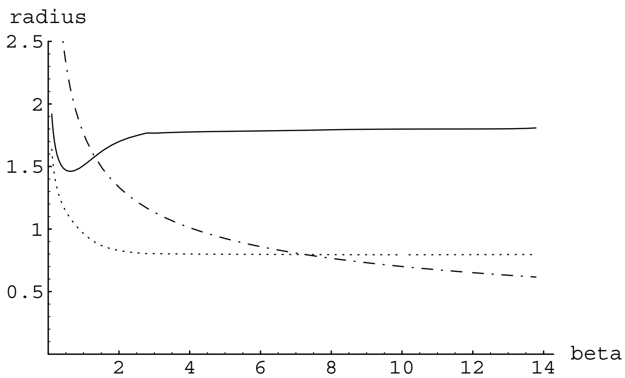

FIG. 3. Range of eigenvalues (radius of the Wigner semicircle) vs β. The upper solid curve is for the scalar fields in the scalar multiplets. The lower dotted curve is for the scalar fields in the gauge multiplet. The dot-dashed curve is the Schwarzschild radius of the black hole. These results were calculated with βF fit to a power law for $\beta > 2.5$.

seriously, then the Gross-Witten transition occurs at $\beta = 14.2$, and is second order (the second derivative of the free energy drops by 0.01 in crossing the transition).

Our prescription for choosing λ begins to break down at about $\beta = 14$, as we find that λ rapidly diverges as β approaches 14. By itself, this is not necessarily a problem: infinite λ simply means that the Wilson loop is uniformly distributed over U(N). But, unfortunately, we do not have a good prescription for continuing past this temperature. Evidently, some of the other gap equations (not just the gap equation for λ) start to break down at this point. Note that this breakdown does not occur until well into the strong coupling regime, as an inverse temperature $\beta = 14$ corresponds to an effective gauge coupling $g_{\text{eff}}^2 = \beta^3 \approx 3 \times 10^3$.

Finally, let us comment on the average "size" of the state. In our approximation the scalar fields $X^i(\tau)$ and $\phi^a(\tau)$ are Gaussian random matrices, and their eigenvalues obey a Wigner semicircle distribution. We can define the size of the state in terms of the quantities

$$\begin{aligned} R_{\text{scalar}}^2 &= \frac{1}{N}\,\text{Tr}\langle [X^i(\tau)]^2\rangle_0\,, \\ R_{\text{gauge}}^2 &= \frac{1}{N}\,\text{Tr}\langle [\phi^a(\tau)]^2\rangle_0\,. \end{aligned} \tag{9}$$

The radius of the Wigner semicircle, given by $2\sqrt{R^2}$, is shown in Fig. 3. Note that the radius stays fairly constant in the region corresponding to the black hole. However, because the superfield formalism we are using does not respect the full SO(9) invariance, the radius measured in the scalar multiplet directions is not the same as the radius measured in the gauge multiplet directions. At $\beta = 14$ we find

$$2R_{\text{scalar}} = 1.81, \qquad 2R_{\text{gauge}} = 0.80\,.$$

This shows that, as expected, the trial action does not respect the underlying SO(9) invariance. Nonetheless, the trial action may provide a useful approximate description of the black hole density matrix in the supergravity regime.

In Fig. 3 we have also plotted the Schwarzschild radius of the black hole $U_0/2\pi = 1.76\beta^{-2/5}$. (Our Higgs fields are related to the radial position by $X = r/2\pi\alpha'$, while Ref. [2] sets $U = r/\alpha'$.) Note that, as the temperature decreases, the Schwarzschild radius becomes much smaller than the radius of the eigenvalue distributions. It seems appropriate to identify the radius of the eigenvalue distributions with the size of the region $U \ll (g_{\text{YM}}^2 N)^{1/3}$ in which ten-dimensional supergravity is valid.

D. K. thanks Rutgers University and New York University for hospitality, and Emil Martinec and Mark Stern for valuable discussions. The work of D. K. is supported by the DOE under Contract No. DE-FG02-90ER40542 and by the generosity of Martin and Helen Chooljian. G. L. thanks the Aspen Center for Physics for hospitality, and Vipul Periwal for useful discussions. The work of G. L. is supported by the NSF under Grant No. PHY-98-02484. D. L. thanks the Aspen Center for Physics, the Abdus Salam International Centre for Theoretical Physics, and the Erwin Schrödinger Institute Program in Duality, String Theory and M-Theory for hospitality during the course of this research, and thanks G. Guralnik, E. Mottola, and K. Rajagopal for helpful discussions. The research of D. L. is supported in part by DOE Grant No. DE-FE0291ER40688-Task A.

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998).
[3] D. Kabat and G. Lifschytz, Nucl. Phys. **B571**, 419 (2000).
[4] D. Kabat and G. Lifschytz, J. High Energy Phys. **9812**, 002 (1998).
[5] D. Kabat and G. Lifschytz, J. High Energy Phys. **9905**, 005 (1999).
[6] D. Kabat, G. Lifschytz, and D. A. Lowe (to be published).
[7] D. Gross and E. Witten, Phys. Rev. D **21**, 446 (1980).
[8] J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).
[9] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).